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Exact Counting of Unlabeled Rigid Interval Posets Regarding or Disregarding Height

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Abstract In this paper, the author gives the exact counting of unlabeled rigid interval posets regarding or disregarding the height by using generating functions. The counting technique follows those introduced in El-Zahar (1989), Hanlon (Trans Amer Math Soc 272:383–426, 1982), Khamis (Discrete Math 275:165–175, 2004). The main advantage of the suggested technique is that a very simple recursive construction of unlabeled rigid interval poset from small ones leads to derive the given generating function for unlabeled rigid interval posets whose coefficients can be easily computed. Moreover, it is proven that the sets of n-element unlabeled rigid interval posets and upper triangular 0–1 matrices with n ones and no zero rows or columns are in one-to-one correspondence. In addition, n-element unlabeled interval posets are counted for $n \geq 1$, using the given generating function for rigid ones. Upper and lower bounds for the number of n-element unlabeled rigid interval posets are given. Also, an asymptotic estimate for the required numbers is obtained. Numerical results for unlabeled interval posets coincide with those given in El-Zahar (1989) and Khamis (Discrete Math 275:165–175, 2004). The exact numbers of n-element unlabeled rigid and general interval posets with and without height k are included, for $1 \leq k \leq n \leq 15$.

Keywords Counting · Poset · Interval poset · Rigid · Height · Generating function · Upper bound · Lower bound · Asymptotic estimate

Mathematics Subject Classification (2000) 06A07 · 05A15

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1 Introduction

The enumeration of various classes of posets is an interesting combinatorial problem. In the exact counting field, there are two different approaches. The first approach is to find the exact number of elements in the considered class. In general, these exact numbers are obtained in the form of closed formulae or by using generating functions, e.g., see [9]. When it is impossible to find the exact counting, researchers look for upper and lower bounds or asymptotic estimates for the required numbers, e.g., see [1] and [4]. The second approach is to find an algorithm for constructing the specified posets and computing the required numbers. The main disadvantage of this approach is that the running time of an algorithm grows more and more rapidly even though the size of the posets is still not large. For example, Heitzing and Reinhold gave a new orderly algorithm to count the number of unlabeled posets on up to 14 elements only (corresponding with a rate of 3000 objects per second on a 450 MHz DEC Alpha), see [8]. In [3], Brinkmann and McKay succeeded in counting the number of posets on up to 16 elements (corresponding with a rate of four million non-isomorphic posets per second on a 1 GHz Pentium III).

The goal of this paper is to count the numbers of unlabeled rigid and general interval posets. The previous studies are very interesting such as: Haxell et.al provided an algorithm, based on a complicated recurrence relation, to produce the first numbers of unlabeled interval posets, [7]. EL-Zahar, [4], introduced the generating function for counting unlabeled interval posets. He followed the technique that has been given by P. Hanlon for counting interval graphs, [6]. Also, the author followed the same technique to count the interval posets according to the height in [9]. EL-Zahar in [4] and khamis in [9] used a recursive description of unlabeled reduced interval posets to derive a pair of functional equations that define the generating function for unlabeled interval posets. In a recent paper [2], Bousquet et al. derived the generating function for unlabeled interval posets via proving correspondences between four combinatorial objects. These objects are unlabeled $(2+2)$-free posets on n elements, certain sequences of n nonnegative integers called ascent sequences, a new class of permutations on n letters, and certain involutions on $2n$ points.

In this paper, the author modifies the previous method that is introduced in [4], by using the unlabeled rigid interval posets. Rigid posets are those posets having only a trivial automorphism. A simple recursive construction method determines the generating function for counting unlabeled rigid interval posets. Furthermore, all results of [4] and [9] can be easily obtained from those types of posets. In addition, the author presents an asymptotic estimate for the number of unlabeled rigid interval posets on n elements. This is done by calculating upper and lower bounds on the required number. The class of interval posets is counted in terms of generating functions for rigid ones in both cases, namely with or without taking the height of a poset into account.

The paper is organized into six sections. In Section 2, the basic definitions, notions, and terminologies which are recalled from [4] and [9], are stated. In Section 3, the derivation of the generating function for unlabeled rigid interval posets is given and then, the required numbers are exactly calculated. Moreover, in this section, we show how to obtain the class of unlabeled interval posets from rigid ones. In Section 4, a bijection between n-element unlabeled rigid interval posets and upper triangular

0–1 matrices with n ones and no zero rows or columns is proven. In addition, some interesting formulae between the numbers of unlabeled rigid and interval posets are introduced. Section 5 contains the details of computing an upper bound, a lower bound, and an asymptotic estimate for the number of unlabeled rigid interval posets on n elements. Section 6 includes an exact height counting for the same underlying posets via using the deduced generating function for unlabeled rigid interval posets. Finally, the Appendix illustrates numerical results for posets in the specified classes with n elements and height k, for $1 \leq k \leq n \leq 15$.

2 Basic Definitions

Let $P = (V, <)$ be a *poset*, where V is a finite non-empty set and $<$ is a partial order defined on V. A subset X of P is called a *chain* if for every $u, v \in X$ either $u < v$ or $v < u$. Also, X is called an *antichain* if for every $u, v \in X$ neither $u < v$ nor $v < u$. The *height* of an element $u \in P$, denoted by $h(u)$, is *the maximum cardinality of a chain in P having u as its maximum element*. The *height* of P is defined as $h(P) = \max(h(u) : \forall\, u \in P)$. The poset of Fig. 1 has the height 5; since the chain $(1 < 2 < 4 < 7 < 11)$ has the maximum cardinality 5.

A poset P is said to be an *interval poset or order* if each element $v \in P$ can be represented by an interval I_v on the real line such that $v < w$ if and only if I_v lies entirely to the left of I_w. It is known that P is an interval poset if and only if it does not contain an induced subposet isomorphic to $2 + 2$, the union of two disjoint 2-element chains, hence P is, also, known by $(2 + 2)$-free poset, [5]. Another characterization of interval posets is that *P is an interval poset if and only if the sets of predecessors (Pred $(u \in P)$=$(v \in P : v < u)$) of the elements of P are linearly ordered by inclusion.*

A poset P is called a *rigid* if it has only a trivial automorphism. P is called a *rigid interval poset* if it satisfies the condition of being interval poset and rigid, see Fig. 1.

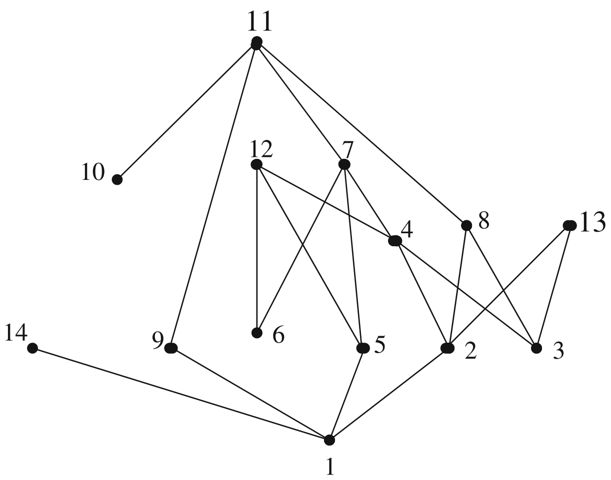


Fig. 1 A rigid interval poset

One aim of this paper is to show that the class of unlabeled interval posets can be constructed and counted from rigid ones instead of from reduced interval posets as mentioned in [4]. Clearly, rigid interval posets are common to researchers more than reduced interval posets. Form [4], we recall that P *is said to be reduced interval poset if There exist no two maximal element of* P *having the same set of predecessors.* According to this definition, rigid interval posets are also considered reduced interval posets but the converse is not true, since a reduced poset which is not rigid, has at least one non-trivial automorphism. So, the class of reduced interval posets contains all rigid interval ones. Fortunately, reduced interval posets can be obtained from small rigid ones by splitting some non-maximal elements within a rigid interval poset in to antichains with suitable sizes. Furthermore, the recursive calculations for required numbers of unlabeled rigid interval posets are much simpler than recursive calculations for reduced interval posets because the numbers of terms of the generating function for unlabeled rigid interval posets are less than those deduced in case of reduced ones. Moreover, the use of rigid interval posets enables us to obtain a bijection between the sets of special upper triangle matrices and unlabeled rigid interval posets.

3 Unlabeled Rigid Interval Posets

In this section, it will be shown that any rigid interval poset P can be constructed from a single element by a sequence of splitting and adding operations which are explained here. An interval poset which is not rigid, can be obtained from rigid poset P by replacing certain elements of P with antichains of size at least 2. The technique follows the same ideas in [4], but the main difference is that the number of n-element unlabeled interval posets is counted by getting the numbers of k-element unlabeled rigid interval posets rather than unlabeled reduced interval posets, for all $k \leq n$.

Now, to enumerate the numbers of unlabeled rigid and general interval posets, we have to recall some definitions, notions, and operations from [4] and [9]. Let P be an interval poset. A maximal element $v \in P$ is called a *chief* element if $h(v) = h(P)$. And then, a maximal element v is called an *assistant* element under the condition of $h(v) < h(P)$. A chief element u of P whose set of predecessors is maximum, is called a *leader* element of P, denoted by $l(P)$. In other words, $l(P)$ is a chief element of P which is larger than all its non-maximal elements. If P is a rigid interval poset, then $l(P)$ is unique and the converse is not true. Figure 2 illustrates a rigid interval poset P having three maximal elements: point 5 is the assistant element, points 6 and 7 are chief elements, but point 7 is the leader of P. However, in Fig. 1, point 11 is the leader and points (12, 13, and 14) are assistant elements.

Given a poset, let y be the weight of a non-maximal element and z be the weight of a maximal one, either chief or assistant. Then, the weight of a poset can be considered as the product of the weights of its elements. In the following two subsections, we use these terminologies to obtain the generating functions for unlabeled rigid and general interval posets.

3.1 Counting Unlabeled Rigid Interval Posets

Suppose P is a rigid interval poset with m maximal elements (chief or assistant) and n non-maximal elements. Then, P has the weight $y^n z^m$. We define the

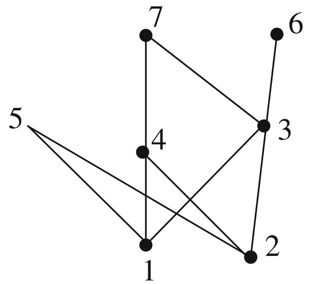


Fig. 2 A rigid interval poset P having three maximal elements

generating function for unlabeled rigid interval posets w.r.t. the weight as $W(y,z) = \sum_{\substack{n,m\geq 0\\ m\neq 0}} w_{nm} y^n z^m$, where w_{nm} denotes the number of unlabeled rigid interval posets having weight $y^n z^m$.

An immediate predecessor, or simply *predecessor*, subposet of a rigid interval poset P is the rigid interval poset obtained from P by deleting $l(P)$ and identifying all pairs of maximal elements, (u_1, u_2), in $P\backslash l(P)$ for which $Pred(u_1) = Pred(u_2)$. Since P is a rigid interval poset, it is impossible to find two elements having the same predecessors and the same successors; thus, when removing $l(P)$, there exist at most two maximal elements having the same predecessors. Let Q be a rigid interval poset with weight $y^n z^m$. To construct all rigid interval posets having Q as their predecessor subposet, we proceed as the following:

1. Adding a new element, u, to Q and joining it as a successor to all non-maximal elements of Q. Then, u is the leader element of the resulting poset say P, and contributes to its weight by z.
2. All non-maximal elements of Q remain non-maximal in P with the same weights as in Q.
3. For a maximal element v of Q, there are three possibilities:
   (a) v *remains* maximal element with weight z, *in* P.
   (b) v becomes a non-maximal element, i.e., $v < u$, with weight y, or
   (c) v splits into v_0 and v_1, where v_0 remains maximal and $v_1 < u$. Clearly in P, v_0 and v_1 have the same predecessors but v_0 is an assistant element and v_1 is covered by $l(P)$.Then, the weight of v is changed by the product the weight of to v_0 and v_1, i.e., by yz.

In terms of generating functions, the above three possibilities for a maximal element v with weight z in Q translates to (z+y+yz) when obtaining weight of new posets.

Since P is required to be rigid, then not all maximal elements of Q remain maximal without splitting in P; otherwise, $l(P)$ and $l(Q)$ have in P the same set of predecessors. Therefore, the generating function for all unlabeled rigid interval posets having Q as their predecessor subposet is given by

$$y^n z\left[z + y + yz\right]^m - y^n z^{m+1}.$$

Figure 3 is an example for applying steps (1)–(3) (with three possibilities) in which the original poset Q has weight y^2z^2. All rigid interval posets $P_1,\ldots,P_8$ that are generated from Q by adding a new maximal element (point 5) and splitting the other two old maximal elements (points 3 and 4) by every possible chances, are illustrated in Fig. 3. Showing that the weight of each of both P_1 and P_2 is y^3z^2, P_3 is y^4z, each of both P_4 and P_5 is y^3z^3, each of both P_6 and P_7 is y^4z^2, and finally, P_8 is y^4z^3.

These weights are equivalent to those produced from the expansion of $y^2z[z + y + yz]^2 - y^2z^3$. In Fig. 3, we use "o" to indicate to the new element or the elements obtained by splitting an original one and by "•" to denote original elements in Q which are still non-maximal in P_i (i = 1,...,8).

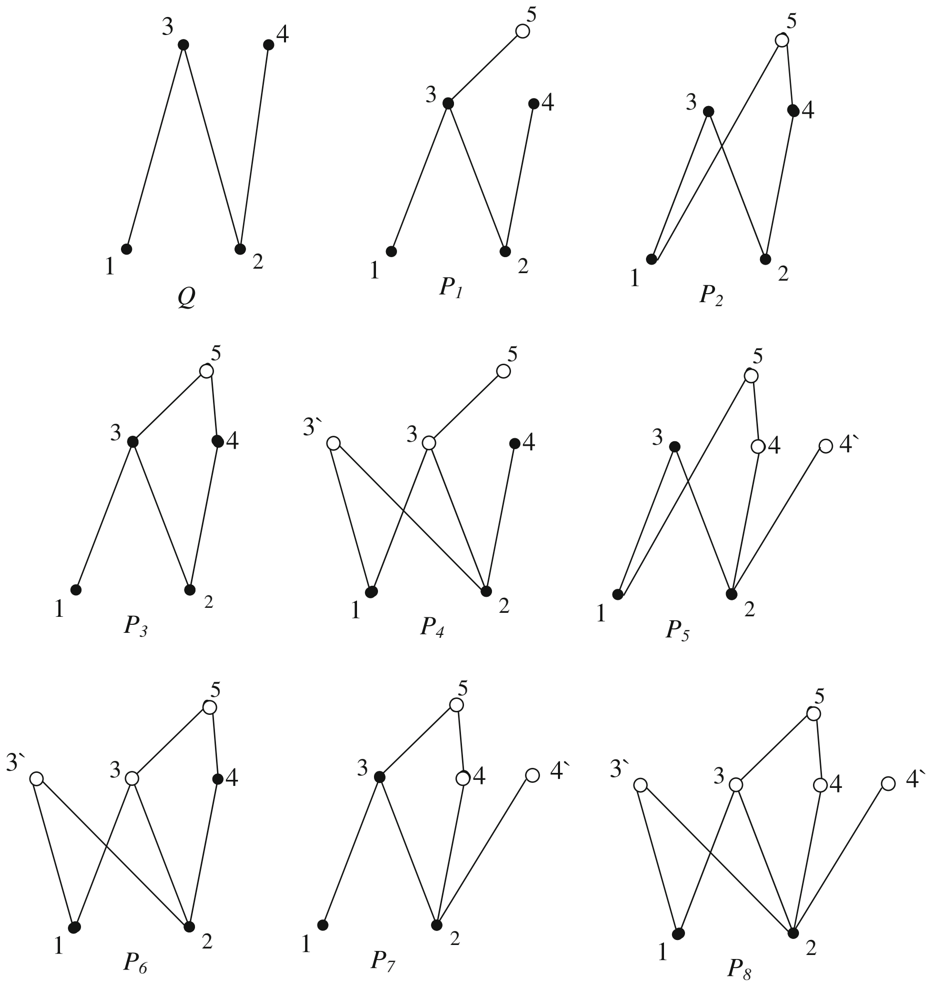


Fig. 3 All rigid interval posets are generated from Q

The above structure for all rigid interval posets leads to derive the generating function for rigid interval posets, $W(y, z)$, which satisfies the following lemma:

Lemma 3.1 *The functional equation that determines $W(y, z)$, is given by*

$$W(y, z) = z + zW(y, z + y + yz) - zW(y, z). \tag{3.1}$$

Proof From the above constructed steps (1)–(3), the R.H.S. of Eq. 3.1 counts the numbers of unlabeled rigid interval posets having n elements that is shown as follows. First, the term z in the R.H.S of the Eq. 3.1 accounts the single-element poset which has neither predecessor nor successor. In fact, any rigid interval poset P having n elements, is obtained from exactly one poset having fewer elements than n. This is guaranteed because the constructed steps produce P by applying only one of the following:

1. Adding a new maximal element larger than at least one maximal element in a poset having cardinality $n - 1$.
2. Adding a new maximal element to a poset Q having cardinality *less than* $n - 1$ after splitting some maximal elements of Q as mentioned in step 3(c). So, the second term of the R.H.S of Eq. 3.1 can be obtained from

$$\sum_{\substack{n+m>0 \\ m\neq 0}} w_{nm} y^n z \left[z + y + yz\right]^m .$$

Evidently, this term takes into account those posets having $l(P) = l(Q)$ produced from the possibility that $l(P)$ does not cover any maximal element of Q. Thus, we should subtract the following

$$\sum_{\substack{n+m>0 \\ m\neq 0}} w_{nm} y^n z^{m+1},$$

which is the third term of the equation. □

Equation 3.1 gives the recursive calculation of w_{nm}. The values of w_{nm} are given in the first table of the Appendix, for $1 \leq n + m \leq 15$. A sample of some terms of $W(y, z)$ is given by

$$W(y, z) = z + yz + \left(yz^2 + y^2z\right) + \left(2y^3z + 3y^2z^2\right) + \left(5y^4z + 9y^3z^2 + 2y^2z^3\right) + \ldots$$

3.2 Counting Unlabeled Interval Posets

In this article, it is presenting how to obtain the generating function for unlabeled interval posets from the introduced generating function for unlabeled rigid ones. Let

$I(x) = \sum_{n\geq 1} i_n x^n$ be the generating function for i_n; the number of unlabeled interval posets of cardinality n. Then, we have the following lemma:

Lemma 3.2 *The generating function, $I(x)$, for counting the numbers of unlabeled interval posets, is*

$$I(x) = W\left(\frac{x}{1-x}, \frac{x}{1-x}\right). \tag{3.2}$$

Proof It is clear that any interval poset can be uniquely obtained from a rigid one by replacing an element in that rigid poset with an antichain. This operation produces a set of elements with the same set of predecessors and the same set of successors. The generating function of all antichains is $x/(1-x)$. Since, all elements within a rigid interval poset are different; thus, substituting antichains of any sizes produces different interval posets. Therefore,

$$\sum_{\substack{n,m>0\\ m\neq 0}} w_{nm}\left(\frac{x}{1-x}\right)^n\left(\frac{x}{1-x}\right)^m,$$

counts all unlabeled interval posets including rigid ones. Hence, the generating function $I(x)$ counts exactly all unlabeled interval posets. □

Equations 3.1 and 3.2 are recursively used to calculate i_n. For example, some coefficients of $I(x)$ are computed and stated as follows.

$$I(x) = x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + \ldots$$

4 Special Upper Triangular Matrices and Unlabeled Rigid Interval Posets

In this section, by knowing the bijection relation between the sets of upper triangular matrices with especial properties and unlabeled interval posets, [2], we establish a bijection relation between the sets of upper triangular 0–1 matrices with n ones and no zero rows or columns and unlabeled rigid interval posets on n elements. First, we give a strategy to constructing the interval poset from its corresponding matrix. This strategy assists to obtain the bijection between the set of upper triangular 0–1 matrices with mentioned properties and the set of unlabeled rigid interval posets.

Finally, these two bijection help to find some interesting formulae for calculating the number of unlabeled interval posets on n elements as a function of the numbers of unlabeled rigid interval posets on k elements, $k < n$, and vice verse.

Let $\mathscr{P}_{\mathrm{n}}$ be the set of unlabeled interval posets on n elements and let $\mathscr{A}_{\mathrm{n}}$ be the set of upper triangular matrices with non-negative integer entries and no zero rows or columns such that sum of all entries is equal to n.

Now, we show how to obtaining $P \in \mathscr{P}_n$ from its associated matrix $A \in \mathscr{A}_n$. Given $A = \left[a_{ij}\right]; 1 \leq i, j \leq k$, where k is the order of A. To construct the associated interval poset P from A, It is easy to apply the following steps:

1. Each entry $a_{ij} = 1$ corresponds to a one element in P.
2. Each entry $a_{ij} = r > 1$ corresponds to an r-element subposet of P, all those elements have the same sets of predecessor and successor elements in P.
3. The element(s) corresponding to the entry $a_{ij} \neq 0$ have successor elements corresponding to $a_{pq} \neq 0$; $j < p \leq q \leq k$ and predecessor elements corresponding to $a_{cd} \neq 0$; $1 \leq c \leq d < i$.
4. All non-zero entries in the same row or the same column in A correspond to incomparable elements in P.

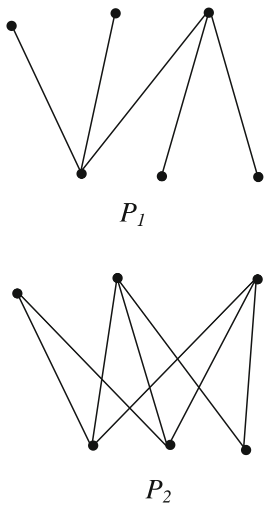
Evidently, non-zero entries in the last column in A correspond to the maximal elements in P. And also, the non-zero entries in the first row correspond to the minimal elements of P (those elements have no predecessors). In addition, the value of entry that lies in first row and last column represents the number of elements in P which have no predecessor and no successor. For example, in Fig. 4, P_1 and P_2 are the associated interval posets to the matrices A_1 and A_2, respectively.

From the suggested correspondence between any matrix and its associated poset, we observe the following:

Observation (1) If it is possible to interchange any two rows and their associated columns in a matrix $A \in \mathscr{A}_n$, the resulting matrix represents a new poset; since this process affects the relation between elements in a poset. Therefore, the poset P which is created by the above process from its associated matrix A, is unique. Indeed, it is worthy to say that any interchange of some columns and the corresponding rows is not always possible. Since, when interchange two rows

Fig. 4 Two different matrices and their associated posets

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$



and the corresponding columns, some entry appears in the lower triangular part of underlying matrix. For illustration, A_1 can be obtained from A_2 as shown in Fig. 4 by interchanging column 1 with column 2 and vice versa, simultaneously, row 2 and row 3. As a result of this, the produced poset P_1 from A_1 is not the same produced poset P_2 from the original matrix A_2.

Observation (2) From step (3), one can directly achieve that the sets of predecessors of elements within the constructed poset P from any matrix $A \in \mathscr{A}_\text{n}$, are linearly ordered by inclusion. That is, P is an interval poset, since the sets of predecessors of elements corresponding to non-zero entries of the same row are identical and the sets of predecessors of non-zero entries of the 1^{st} row $\subseteq$ those corresponding to the 2^{nd} row $\subseteq$ up to the last row.

Observation (3) It is clear that if a matrix $A \in \mathscr{A}_\text{n}$ contains at least one entry having value $r > 1$, then the associated poset P has a subposet of r elements whose sets of predecessors and successors are the same. In this case, P is an interval poset which is not rigid.

As a consequence of the last observation, we conclude the following lemma:

Lemma 4.1 *Let $\mathscr{Q}_\text{n}$ be the set of unlabeled rigid interval posets on n elements and $\mathscr{B}_\text{n}$ be the set of upper triangular* 0–1 *matrices with n ones and no zero rows or column. For $n \geq 1$, $\mathscr{Q}_\text{n}$ and $\mathscr{B}_\text{n}$ are in one-to-one correspondence.*

Proof The proof is a straight forward from the facts that $\mathscr{Q}_\text{n} \subset \mathscr{P}_\text{n}$ and $\mathscr{B}_\text{n} \subset \mathscr{A}_\text{n}$, and Observation 3. Exclude those matrices which are belonging to $\mathscr{A}_\text{n} \setminus \mathscr{B}_\text{n}$. Then, for each excluded matrix remove its associated poset from $\mathscr{P}_\text{n}$. Since, the excluded matrices are those matrices having at least one entry with value greater than one; therefore, the removed associated posets are those posets having some suposets of at least two elements possessing the same sets of predecessors and successors. Clearly, since the relation between $\mathscr{A}_\text{n}$ and $\mathscr{P}_\text{n}$ is a bijection, [2], each excluded matrix associates only one excluded poset. Equivalently, each remaining upper triangular matrix with n ones and no zero rows or columns whose entries are 0 or 1 only corresponds to exactly one of the remaining posets. Of course, each remaining poset is rigid interval poset, since there are no two elements within the poset having the same sets of predecessors and successors. This completes the proof. □

Figure 5 illustrates an instance element of $\mathscr{B}_\text{n}$ and its associated poset belonging to $\mathscr{Q}_\text{n}$.

Note The bijection between $\mathscr{Q}_\text{n}$ and $\mathscr{B}_\text{n}$ is original and the Encyclopedia of integer sequences is not including it, [10].

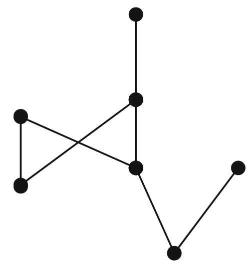
Some interesting formulae In a recent paper [2], the authors introduced the functional equation of $I(x)$ which is given by

$$I(x) = \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - (1 - x)^i\right). \tag{4.1}$$

Fig. 5 A rigid interval poset and its corresponding matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A



P

This series counts the number of permutations on n letters, the number of nonisomorphic intervals on n unlabeled points, and certain nonnegative integers called ascent sequences of length n, [2]. Furthermore, $I(x)$ counts certain involutions or regular linearized chord diagrams on $2n$ points, see for more details [11] and [12]. Also, $I(x)$ counts the number of upper triangular matrices with nonnegative integer entries and without zero rows or columns such that sum of all entries is equal to n. These numbers and the above expression of $I(x)$ occur in the Encyclopedia of integer sequences as sequence A022493 [10].

From ([10], A138265), we have the functional equation of generating function $W(x)$ for w_n which is the cardinality of $\mathcal{B}_n$, $n \geq 1$. In the following, we ensure that $W(x)$ is, also, counting the cardinality of $\mathcal{Q}_n$, $n \geq 1$. Intuitively, this is a logical result as mentioned in Lemma 4.1.

Lemma 4.2 *The functional equation for determining the number of unlabeled rigid interval posets on n elements, $n \geq 1$, is given by*

$$W(x) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1 - \frac{1}{(1+x)^i}\right). \tag{4.2}$$

Proof It is easy to deduce $W(x)$ from $I(x)$ by taking into consideration that the expansion of $(x/(1-x) - x)$ produces all k-element subposets, $k > 1$, whose elements having the same sets of predecessors and successors. We are primarily concerned with the term $(1-x)^i$ in the R.H.S. of Eq. 4.1. This term can be rewritten as

$$(1-x)^i = \frac{1}{(1-x)^{-i}},$$

$$= \frac{1}{\left(1 + x + x^2 + x^3 + \ldots\right)^i}$$

Then, we obtain

$$I(x) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1 - \frac{1}{\left(1 + x/(1-x)\right)^i}\right).$$

Since, splitting any element within a rigid poset into k-element antichain, $k > 1$, this produces interval posets, the reverse operation is to identify those elements having the same sets of predecessors and successors to unique element. By successively making the replacement of subposets within a poset which have more than one element, with one representative element, the elements of the resulting poset are different w.r.t. predecessors and successors. Actually, applying this operation on the set of interval posets leads to obtain the set of rigid interval ones. In a mathematical manipulation of generating functions, the replacement operation is equivalent to replace $x/(1-x)$ by x in the R.H.S. of the above expression of $I(x)$. Therefore, we obtain the functional equation for evaluating unlabeled rigid interval posets. □

Finally, we show how the introduced generating functions for unlabeled rigid and interval poets permit us to evaluate the coefficients of $I(x)$ in terms of the coefficients of $W(x)$ and vice versa. As given in Section 3.2, $I(x)$ can be expressed as a functional equation of $W(x)$. This yields via replacing x by $x(1-x)$ as we follow.

$$I(x) = W\left(\frac{x}{1-x}\right). \tag{4.3}$$

Equating the coefficients of both sides of Eq. 4.3, one can consequently conclude the following:

$$i_n = \sum_{k=0}^{n-1} \binom{n-1}{k} W_{k+1}, \tag{4.4}$$

which can be written as

$$i_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \times A138265\,(k+1),$$

and this is an original formula.

Additionally, we have

$$w_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n-1}{k-1} i_k. \tag{4.5}$$

This is already given by

$$w_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n-1}{k-1} \times A022493\,(k).$$

As given in ([10], A138265). For example of integer sequences i_n and w_n are stated, respectively, as follows.

$$(1, 1, 2, 5, 15, 53, 217, 1014, 5335, \ldots) \text{ and } (1, 1, 1, 2, 5, 16, 61, 271, 1372, \ldots).$$

5 Asymptotic Formulae for Rigid Interval Poset

This section presents how to obtain asymptotic estimate for w_n; the number of unlabeled rigid interval posets on n elements. As a result of the bijection between unlabeled interval posets and certain upper triangular matrices, upper and lower bounds on rigid interval posets are found. Let J_n be the number of upper triangular

0−1 matrices with n ones, no zero rows or columns, and no zeros on the main diagonal, and G_n be the number of upper triangular 0−1 matrices with n ones.

To obtain a lower bound on w_n, we sufficiently count all upper triangular 0−1 matrices having no zero on the main diagonal. This type of matrices represents the class of unlabeled rigid interval N-free posets on n elements (a poset P is called an N-free if it does not contain subposet isomorphic to Q of Fig. 3), for details see [1]. In order to determine those matrices of order, $k > 0$, first we put k ones on the main diagonal. Then, for the remaining $n-k$ ones, choosing $n-k$ places out of $k(k-1)/2$ places. Therefore, we have $\binom{k(k-1)/2}{n-k}$ matrices, and obtain the following:

Lemma 5.1 *A lower bound on w_n is given by*

$$J_n = \sum_{k=1}^{n} \binom{k(k-1)/2}{n-k}. \tag{5.1}$$

In the similar way, we get an upper bound on w_n. This is done by counting all upper triangular 0−1 matrices of order k having n ones and no other restrictions. So, one can easily obtain the number of these matrices which is given by $\binom{k(k+1)/2}{n}$ ways. Then, we have

Lemma 5.2 *An upper bound on w_n is given by*

$$G_n = \sum_{k=1}^{n} \binom{k(k+1)/2}{n}. \tag{5.2}$$

To complete our purpose of finding an asymptotic estimate on w_n, we combine upper and lower bounds which leads to the following result:

Theorem 5.1 *An asymptotic estimate on w_n is determined by*

$$w_n = 2^{n\lg(n)+o(n\lg(n))}. \tag{5.3}$$

Proof To prove Formula 5.3, we combine the lower and upper bounds given in the above Lemmas 5.1 and 5.2 as follows.

$$J_n < w_n < G_n$$

Then, it is sufficient to estimate both J_n and G_n. Assume n is large and put

$$m = \frac{n}{\lg(n)+2}.$$

Now, Eq. 5.1 implies that $J_n > \binom{m^2/2}{n-m}$. Using striling, one easily deduces that if a>>b>>1, then

$$\lg\binom{a}{b} \cong b\lg(a/b),$$

$$\lg(J_n) > n\lg(n). \tag{5.4}$$

Similarly, by the same assumption, Eq. 5.2 implies that $G_n < n\binom{(n+1)^2}{n}$. Consequently,

$$\begin{aligned} lg(G_n) &< lg(n) + n\, lg(2n) \\ &< n\, lg(n) + n + n\, lg(2). \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we have $\lg(J_n) < lg(w_n) < lg\ (G_n)$. This gives an asymptotic logarithm (base 2) on w_n,

$$lg(w_n) = n\, lg(n) + o(n\, lg\ (n)),$$

this completes the proof. □

6 Height Counting of Unlabeled Rigid Interval Posets

In this section, an enumeration of unlabeled rigid interval posets according to the height is demonstrated. The derived function for the height counting of unlabeled interval posets in [9] can be easily deduced from the rigid case. First, we restate the following notions from [9].

As given in Section 3, in a poset P, let y be the weight of a non-maximal element, z be the weight of an assistant element, and w be the weight of a chief element. Assume that P is a rigid interval poset with height k and having n non-maximal elements, r assistant elements, and s chief elements. Then, the weight of P which is viewed as the product of the weights of its elements, is $y^n z^r w^s$. Define the generating function for unlabeled rigid interval posets w.r.t. their weights and heights as

$$D(y, z, w, h) = \sum_{\substack{n, r \geq 0 \\ s, k \geq 1}} d_{nrsk} y^n z^r w^s h^k,$$

where d_{nrsk} denotes the number of unlabeled rigid interval posets with weight $y^n z^r w^s$ and height k. As in Section 3, we describe how rigid interval posets are built from small ones taking the height into account.

Let Q be a rigid interval poset with $h(Q) = k$. To obtain all unlabeled rigid interval posets P having Q as their predecessor subposet, we proceed as the following:

1. Adding a new element $l(P)$, as defined in Section 3, larger than all non-maximal element of Q; $l(P)$ will have weight w.
2. All non-maximal elements of Q remain non-maximal in P, i.e., their weights remain the same in P.
3. For an assistant element v of Q with weight z, there are three possibilities:
   (a) v remains assistant element in P with weight z,
   (b) v becomes a non-maximal element in $P(v < l(P))$ with weight y, or
   (c) v splits into v_0 and v_1, where v_0 remains assistant element in P and $v_1 < l(P)$. The sets of predecessors of v_0 and v_1 are identical but v_0 has no successor; however, v_1 has one successor $l(P)$. Then, the weight of v is changed by the product of the weights of v_0 and v_1, i.e., by yz.

To obtain all possibilities, we replace the weight z *in* Q by $z + y$+yz to obtaining new posets.

4. For a chief element v of Q with weight w, there are two cases:
   (a) $h(P) = h(Q)$. In this case, the only possibility for v is to remain a chief element without splitting, so, the weight is still the same.
   (b) $h(P) = h(Q) + 1$. Here, there are three possibilities:
      (b.1) v remains a maximal element in P. Since $h(v) < h(P)$, then v becomes an assistant element and its weight is altered to z in the weight of P,
      (b.2) v becomes a non-maximal element in P. Replace the weight w by y, or
      (b.3) v splits into v_0 and v_1, where v_0 is an assistant maximal element in P and $v_1 < l(P)$. v_0 and v_1 have the same set of predecessors and differ in the sets of successors. So, replace the weight of v by yz.

In conclusion, the weight of$v$ is not changed if $h(P) = h(Q)$ and becomes $z + y + yz$ if $h(P) = h(Q) + 1$. Note that $h(P) = h(Q) + 1$ if and only if $l(P)$ covers at least one chief element v of Q or a non-maximal element arising from the splitting of a chief element. Otherwise, $h(P) = h(Q)$.

Now, assume that Q has weight $y^n z^r w^s$ and height k. Let $D_1(y, z, w, h)$ and $D_2(y, z, w, h)$ be the weight enumerators of unlabeled rigid interval posets having Q as their predecessor subposet and of height $k + 1$ and k, respectively. We have the following two results.

Lemma 6.1

$$D_1(y, z, w, h) = y^n (z + y + yz)^r \left((z + y + yz)^s - z^s\right) wh^{k+1}.$$

Proof Let P be a rigid interval poset with height $k + 1$ and having Q as its predecessor subposet. Since $h(P) > h(Q)$, then the only chief element of P is $l(P)$. Furthermore, not all chief elements of Q remain maximal without splitting in P, since otherwise, we would have $h(P) > h(Q)$. This explains the subtraction of the term z^s in the substitution for w^s. The substitutions for y^n and z^r follow, respectively, from steps (2) and (3) mentioned above. □

Lemma 6.2

$$D_2(y, z, w, h) = y^n \left((z + y + yz)^r - z^r\right) w^{s+1} h^k.$$

Proof Suppose that P is a rigid interval poset with height k and having Q as its predecessor subposet. Since P is rigid, then not all assistant elements of Q remain assistant without splitting in P. Otherwise, $l(P)$ and $l(Q)$ would have the same set of predecessors. Therefore, the term z^r is replaced by $(z + y + yz)^r - z^r$. The remaining terms are straightforward. □

As a result of the above two lemmas, $D(y, z, w, h)$ satisfies the following equation:

Theorem 6.1

$$D\,(y, z, w, h) = wh + wh(D(y, (z + y + yz), (z + y + yz), h) - D(y, (z + y + yz), z, h)) + w(D(y, (z + y + yz), w, h) - D(y, z, w, h)). \tag{6.1}$$

Proof The term wh on the R.H.S. of Eq. 6.1 accounts for the single-element poset which has neither predecessor nor successor. Every other rigid interval poset has a unique predecessor and therefore, its weight appears exactly once in $\Sigma(D_1(y, z, w, h) + D_2(y, z, w, h))$, where the summation is taken over all $n, r \geq 0$ and $s \geq 1$. The required result now follows. □

Let $G(x, h) = \sum_{1 \leq k \leq n} g_{nk} x^n h^k$, be the generating function for g_{nk}; the number of unlabeled interval posets having n elements and height k. Each interval poset P is obtained from a unique rigid interval poset Q via replacing some elements of Q by antichains. The generating function of all antichains is $x/(1 - x)$. Therefore, we have

Corollary 6.1

$$G(x, h) = D\left(\frac{x}{1-x}, \frac{x}{1-x}, \frac{x}{1-x}, h\right). \tag{6.2}$$

Equation 6.1 can be used recursively to calculate the coefficients d_{nrsk}. Then from Eq. 6.2, one can recursively calculate g_{nk}. Finally, we state some terms of the above generating functions.

$$D(y, z, w, h) = wh + ywh^2 + \left(yzwh^2 + y^2wh^3\right) + \left(y^2w^2h^2 + \left(y^3w + 2y^2zw\right)h^3 + y^3wh^4\right) + \ldots$$

$$G\,(x, h) = xh + x^2\left(h + h^2\right) + x^3\left(h + 3h^2 + h^3\right) + x^4\left(h + 7h^2 + 6h^3 + h^4\right) + \ldots$$

The values of $d_{pk} = \sum_{p=n+r+s} d_{nrsk}$ and g_{pk} are included in Tables 2 and 3, respectively, of the Appendix, for $1 \leq k \leq p \leq 15$.

7 Conclusion

In this paper, a recursive description of rigid interval posets gives a functional equation for their generating function (Section 3). The modification for counting underling posets taking into account the height of posets is given (Section 6). Moreover, it is shown how the generating function for all unlabeled interval posets can be reconstructed from the generating function for rigid ones (Section 3.2). We have proven that there exists a one-to-one correspondence between n-element unlabeled rigid interval posets and upper triangular 0–1 matrices with n ones and no zero rows or columns which have already been enumerated in ([10], A138265). In addition, some interesting formulae related to i_n's and w_n's are stated (Section 4). Finally, an asymptotic estimate for unlabeled rigid interval posets having n elements is obtained (Section 5).

For a future work, we would like to enumerate two-dimensional posets and N-free posets regarding the height by using the principle of generating functions.

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Appendix

Here, we recorded some numerical results using recurrence formulae given above. The numbers of n-element unlabeled rigid interval posets and interval posets for $n \leq 15$ are calculated in both cases regarding or disregarding of the height. Any researcher can use the introduced formulae and (symbolic) mathematical software packages such as Maple or Mathematica to get more results. But, here, we give these numbers as a sample for any comparison between the previous results. The numbers of Tables 1, 2, and the last row of 3 are new. Furthermore, the other numbers of Table 3 coincide with those given in [9].

Table 1 The values of w_{nm}; the number of unlabeled rigid interval posets with n non-maximal and m maximal elements $1 \leq n+m \leq 15$

n \ m	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	1	3	2	0	0	0	0	0
3	2	9	13	6	0	0	0	0
4	5	32	72	69	24	0	0	0
5	16	132	409	605	432	120	0	0
6	61	623	2480	5016	5498	3120	720	0
7	271	3314	16222	41955	62626	54370	25560	5040
8	1372	19628	114594	363123	690935	814690	584580	
9	7795	128126	872336	3287492	7644536	11464099		
10	49093	914005	7132352	31272370	86299772			
11	339386	7074517	62408068	313016308				
12	2554596	59050739	582316795					
13	20794982	528741491						
14	182010945							

Note that $w_{01} = 1$

Table 2 The values of d_{nk}; the number of n-element unlabeled rigid interval posets of height k, $1 \leq k \leq n \leq 15$

K \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2		1	1	1	1	1	1	1	1	1	1	1	1	1	1
3			1	3	8	20	51	134	367	1048	3119	9655	31024	103269	355435
4				1	6	29	128	555	2427	10864	50145	239581	1186941	6101070	32532429
5					1	10	75	500	3190	20129	127912	827053	5474614	37236376	260803006
6						1	15	160	1480	12835	108308	907923	7654921	65420100	569555033
7							1	21	301	3661	41041	441553	4664814	49075731	518819733
8								1	28	518	7980	111671	1482110	19123146	243593132
9									1	36	834	15828	26944	4305582	66302132
10										1	45	1275	29175	592230	11182368
11											1	55	1870	50710	1208020
12												1	66	2651	83996
13													1	78	3653
14														1	91
15															1
Total	1	1	2	5	16	61	271	1372	7795	49093	339386	2554596	20794982	182010945	1704439030

Table 3 The values of g_{nk}; the numbers of n-element unlabeled interval posets of height k, $1 \leq k \leq n \leq 15$

k \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383
3			1	6	26	100	366	1317	4743	17275	64029	242371	938741	3723210	15123393
4				1	10	69	412	2305	12551	67933	370168	2046980	11546918	66665327	394753250
5					1	15	150	1270	9920	74525	551232	4072130	30322587	228997374	1761229422
6						1	21	286	3236	33301	325860	3109628	29395997	278111527	2651670801
7							1	28	497	7210	93926	1151416	13644127	158939927	1840493674
8								1	36	806	14540	232891	3477454	49791316	695949962
9									1	45	1239	27147	522840	9308502	157771076
10										1	55	1825	47665	1084540	22639660
11											1	66	2596	79596	2109151
12												1	78	3587	127480
13													1	91	4836
14														1	105
15															1
Total	1	2	5	15	53	217	1014	5335	31240	201608	1422074	10886503	89903100	796713190	7541889195

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