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M. M. Nassar[a]; S. M. Khamis[a]; S. S. Radwan[b]
[a] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt [b] Department of Mathematics, Faculty of Science, Al Azhar University, Cairo, Egypt

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On Bayesian sample size determination

M.M. Nassar[a]*, S.M. Khamis[a] and S.S. Radwan[b]

[a]*Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt;*
[b]*Department of Mathematics, Faculty of Science, Al Azhar University, Girls Section, Nasr City, Cairo, Egypt*

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Three Bayesian methods are considered for the determination of sample sizes for sampling from the Laplace distribution – the distribution of time between rare events – with a normal prior. These methods are applied to the sizing of aircraft mid-air collisions in a navigation system or large flight path deviations of aircraft in air traffic management scenarios. A computer program handles all computations and gives a good insight about the best suggested method.

Keywords: kth average coverage criterion; kth average length criterion; worst outcome criterion

1. Introduction

The determination of the sample size is often an important step in the planning of a statistical study, and it is usually a difficult one; however, it is useful to find the suitable sample size prior to sampling, mainly due to time and cost constraints. Approaching the sample size question as a decision problem, we argue that a solution needs to address the choice of the sample size as a part of a larger decision problem, involving both the sample decision before carrying out the experiment and the later decision about the multiple comparisons once the data have been collected.

Many authors discussed this problem: Adcock [1,3–5] has considered multinomial experiments which of course include the binomial as a special case. Pham-Gia [20] and Pham-Gia and Turkkan [21] deal with the parameter of a binomial distribution following a beta prior and present the complete and exact solutions for determining the sample size. Joseph *et al.* [13] determine the sample size for a binomial proportion by using three different Bayesian approaches. The difference between two binomial proportions has been studied by Joseph *et al.* [12]. Adcock [2] has treated the case of a normally distributed response variable. Gelfand and Wang [9] have been concerned with the selection of sample size by adopting a screening criterion. Pezeshk [19] has reviewed some key techniques of Bayesian methods of sample size determination. In their article, Inoue *et al.* [11] provide a simple but general framework for investigating the relationship between the Bayesian and frequentist approaches. In clinical trials, the determination of sample

*Corresponding author. Email: m_nassar_1999@yahoo.com

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size under normal likelihoods has been discussed by Sahu and Smith [23], where at substantive testing stage of financial audit, they concluded that normality is not an appropriate assumption. Bayesian approaches have been formulated by Stamey *et al.* [26] to the problem of determining sample size for Bayesian interval estimators of a predetermined length for a single Poisson rate, for the difference between two Poisson rates and for the ratio of two Poisson rates. M'Lan *et al.* [16] investigate Bayesian sample size determination and the control-to-case ratio for case-control studies, when interval estimation is the goal of the eventual statistical analysis. A Bayesian simulation approach has been developed by Stamey and Gerlach [25] for determining the sample size required for a case-control study with misclassified data using an average posterior variance criterion considering both fixed cost and fixed variance procedures. Willan [28] has proposed a model for expected total profit that includes consideration of per-patient profit, disease incidence and other functions. Several new results have been presented by M'Lan *et al.* [17] for estimating binomial proportions. Also, Nassar *et al.* [18] treated the problem of determining the sample size in the case of a geometric response variable.

The purpose of our paper is to determine the sample size in a Bayesian analysis of a Laplace response variable by using different criteria – discussed in Section 3 - which are based on highest posterior density (HPD). In Section 4, we carry out the study to derive closed form expressions for the optimal sample size focusing on samples from the Laplace distribution when the prior is normal. The discussion in Section 5 includes specific algorithms that are used to evaluate the proposed diagnostics.

2. Examples and applications

Over the last decade, there has been a marked interest in models following the Laplace distribution especially in finance, engineering, astronomy and environmental sciences [7,14,15].

A motivating application – as discussed by Hsu [10] and Compos and Marques [8] – explains the use of the Laplace distribution in a navigation system which consists of specific route configurations in a navigation area or stated traffic control scheme, certain navigation devices and appropriate types of aircraft. An important question about any navigation system is 'How safe the system is in terms of the expected number of collisions per million hours of traveling time at a desired traffic density?'. The answer, which depends on the distribution of position errors, may call for decisions involving great amounts of cost, time and effort.

Since a position error is defined as the difference between the nominal and actual position of an aircraft, the distribution of position errors shows a strong tendency, especially in the tail region, to follow the Laplace distribution given by the following density function:

$$f(x|\theta,\beta) = \begin{cases} \dfrac{1}{2\beta}\exp\left[-\left(\dfrac{x-\theta}{\beta}\right)\right], & \in x \geqslant \theta, \\ \dfrac{1}{2\beta}\exp\left[\dfrac{x-\theta}{\beta}\right], & x < \theta, \end{cases} \tag{1}$$

where $\beta > 0$ is a scale parameter and the shift parameter θ follows the normal density

$$g(\theta) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2\right], \quad -\infty < \theta, \quad \mu < \infty, \quad \sigma > 0. \tag{2}$$

Trying to predict the best location to drill for oil, knowing the positions of existing drill sites in the region and their yields, one needs to rely on a model of the geological processes that gave rise to the yields in those sites for which one has data, in order to be able to predict the best location and the depth of the well to be drilled. These data from an oil well might be measurements of the type of oil found, the geographical location of the well, its operational age, and a host of other measurable quantities regarding its local geological characteristics.

So *et al.* [24] have shown that in studies of financial time series, the distribution of financial returns tends to be fat-tailed following the Laplace distribution. They showed that the distribution is a promising candidate for simulating tail events of returns data: events that lead to extremely large gains or losses.

3. Bayesian criteria for sample size

The three criteria for Bayesian sample size determination based on HPD intervals will be presented in this section.

Let θ be the parameter of our interest, Θ the parameter space of θ, and $g(\theta)$ the prior distribution of θ. We assume that the experiment under consideration provides the data $x = (x_1, x_2, \ldots, x_n)$, where the components of x are exchangeable and belong to the data space X, and n is the sample size.

The predictive distribution of x, also known as the pre-posterior marginal distribution of the data, is given by

$$f(x) = \int_{\Theta} f(x|\theta)g(\theta)\,\mathrm{d}\theta \tag{3}$$

and the posterior distribution of θ given the data x is given by

$$\psi(\theta|x, n) = \frac{f(x|\theta)g(\theta)}{f(x)} \tag{4}$$

where $f(x|\theta)$ is the likelihood of the data.

Since θ is a one-dimensional real-valued parameter and $\psi(\theta|x, n)$ is unimodal, (a, b) is an HPD interval if and only if $\psi(\theta_1|x, n) \geqslant \psi(\theta_2|x, n)$ for all θ_1 in (a, b) and all θ_2 not in (a, b). Under these conditions, an experimenter typically would specify that θ should fall in an HPD interval of length ℓ with probability $1 - \alpha$, that is,

$$\int_a^b \psi(\theta|x, n)\,\mathrm{d}\theta = 1 - \alpha.$$

However, the fact that the posterior distribution depends on x whose uncertainty must be eliminated leads to the consideration of the following criteria – as proposed by Joseph *et al.* [13] and generalised by M'Lan *et al.* [16] – in the determination of the minimum sample size.

3.1 *kth Average coverage criterion*

Let k be a positive integer. We define the kth average coverage criterion (ACC_k) sample size to be the minimum n satisfying

$$\left[\int_{-\infty}^{\infty} \left\{\int_{a(x,n)}^{a(x,n)+\ell} \psi(\theta|x, n)\,\mathrm{d}\theta\right\}^k f(x)\,\mathrm{d}x\right]^{1/k} \geqslant 1 - \alpha, \tag{5}$$

where $f(x)$ is given by Equation (3), $\psi(\theta|x, n)$ by Equation (4) and $a(x, n)$ the lower limit of the HPD credible set of length ℓ for the posterior density $\psi(\theta|x, n)$. This yields the smallest n for which HPD intervals of length ℓ provide an average posterior coverage of at least $1 - \alpha$, where the average is with respect to the marginal distribution under the L_k norm, that is, the kth marginal moment of

$$\delta^*_\ell(x|n) = \int_{a(x,n)}^{a(x,n)+\ell} \psi(\theta|x,n)\,\mathrm{d}\theta. \tag{6}$$

The ACC$_k$ extends the special case, the average coverage criterion, ACC$_1$ = ACC first proposed by Joseph *et al.* [13].

3.2 *kth Average length criterion*

Conversely, the kth average length criterion (ALC$_k$) fixes the desired posterior coverage at $1-\alpha$. The lengths of all HPD intervals are averaged with respect to the marginal distribution $f(x)$ under the L_k norms. One then seeks the minimum n such that

$$\left\{\int_{-\infty}^{\infty} \left[\ell'_{1-\alpha}(x,n)\right]^k f(x)\,\mathrm{d}x\right\}^{1/k} \leqslant \ell, \tag{7}$$

where ℓ is the prespecified average length and $\ell'_{1-\alpha}$ the length of an HPD interval of coverage $1-\alpha$. Again, the ALC$_k$ includes as special case, the average length criterion, ALC$_1$ = ALC, first introduced by Joseph *et al.* [13].

3.3 *Worst outcome criterion*

Both of the above criteria, ACC$_k$ and ALC$_k$, are based on averages over all samples, and give no guarantee for any particular observed data x. A conservative approach is to ensure a maximum length of ℓ and a minimum coverage probability of $(1-\alpha)$, over all possible data x. Thus, rather than averaging, we choose the minimum n such that

$$\inf_{x\in X}\{\delta^*_\ell(x|n)\} \geqslant 1-\alpha, \tag{8}$$

where both ℓ and α are fixed in advance.

4. Bayesian sample size for Laplace Sampling

As an application of the criteria given in Section 3, consider the parameter θ as a random variable with a normal prior distribution given by Equation (2). Also, consider Laplace sampling for x, where $f(x_i|\theta,\beta),\ i=1,2,\ldots,n$, is given by Equation (1). Probably, some of the components of the data x, say $k \leqslant n$, fall in the region $(-\infty,\theta)$, and the rest $(n-k)$ components fall in the other region $[\theta,\infty)$. Arranging those data points in ascending order and using the order statistics for the data points, we obtain

$$x_{(1)}, x_{(2)}, \ldots, x_{(k)} < \theta$$

and

$$x_{(k+1)}, x_{(k+2)}, \ldots, x_{(n)} \geqslant \theta.$$

The likelihood function L of the data is then, for $x_i < \theta$, given by

$$\begin{aligned} L &= \frac{n!}{(n-k)!}\prod_{i=1}^{k} f(x_i|\theta)[1-F(x_{(k)})]^{n-k} \\ &= \frac{n!}{(n-k)!}\left(\frac{1}{2}\right)^n\left(\frac{1}{\beta}\right)^k \exp\left[(n-2k)\frac{\theta}{\beta}\right]\exp\frac{1}{\beta}\left[\sum_{i=1}^{k-1}x_{(i)}-(n-k-1)x_{(k)}\right], \end{aligned} \tag{9}$$

whereas, for $x_i \geqslant \theta$,

$$L = \frac{n!}{k!}[F(x_{(k+1)})]^k \prod_{i=k+1}^{n} f(x_i|\theta)$$
$$= \frac{n!}{k!}\left(\frac{1}{2}\right)^n\left(\frac{1}{\beta}\right)^{n-k} \exp\left[(n-2k)\frac{\theta}{\beta}\right] \exp\frac{1}{\beta}\left[(k-1)x_{(k+1)} - \sum_{i=k+2}^{n} x_{(i)}\right]. \quad (10)$$

Thus, the preposterior marginal distribution of x is given by

$$f(x) = \begin{cases} \Phi(x)\dfrac{n!}{(n-k)!}\left(\dfrac{1}{2}\right)^n\left(\dfrac{1}{\beta}\right)^k \exp\left[\left(\dfrac{n-2k}{\beta}\right)\left(\mu + \dfrac{(n-2k)\sigma^2}{2\beta}\right)\right] \\ \quad \times \exp\left[\dfrac{1}{\beta}\left(\sum_{i=1}^{k-1} x_{(i)} - (n-k-1)x_{(k)}\right)\right], \quad -\infty < \theta < x, \\ [1-\Phi(x)]\dfrac{n!}{k!}\left(\dfrac{1}{2}\right)^n\left(\dfrac{1}{\beta}\right)^{n-k} \exp\left[\left(\dfrac{n-2k}{\beta}\right)\left(\mu + \dfrac{(n-2k)\sigma^2}{2\beta}\right)\right] \\ \quad \times \exp\left[\dfrac{1}{\beta}\left((k-1)x_{(k+1)} - \sum_{i=k+2}^{n} x_{(i)}\right)\right], \quad x < \theta < \infty, \end{cases} \quad (11)$$

where

$$x = \frac{x_{(k)} + x_{(k+1)}}{2} \quad (12)$$

and

$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp -\frac{1}{2\sigma^2}\left[\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right]^2 \mathrm{d}\theta. \quad (13)$$

The posterior distribution function of θ is then given by

$$\psi(\theta|n, \mu, \sigma, \beta)$$
$$= \begin{cases} \dfrac{[\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{1}{2\sigma^2}\left(\theta - \left(\mu + \dfrac{\sigma^2}{\beta}(n-2k)\right)\right)^2\right], \quad -\infty < \theta < x, \\ \dfrac{[1-\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{1}{2\sigma^2}\left(\theta - \left(\mu + \dfrac{\sigma^2}{\beta}(n-2k)\right)\right)^2\right], \quad x < \theta < \infty. \end{cases} \quad (14)$$

Now, the mean and variance of the prior and posterior distributions are

$$\mu_{\text{prior}} = \mu, \qquad \text{var}_{\text{prior}} = \sigma^2, \quad (15)$$

$$\mu_{\text{post}} = \mu^*\left[\frac{\Phi(x^*)}{\Phi(x)} + \frac{1-\Phi(x^*)}{1-\Phi(x)}\right] + \sigma^2\varphi(x^*)\left[\frac{1}{1-\Phi(x)} - \frac{1}{\Phi(x)}\right], \quad (16)$$

where

$$\mu^* = \mu + \frac{\sigma^2}{\beta}(n-2k),$$
$$x^* = \frac{x-\mu}{\sigma}, \quad x = \frac{x_{(k)} + x_{(k+1)}}{2}, \quad (17)$$
$$\varphi(x^*) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^{*2}\right], \quad -\infty < x^* < \infty.$$

Also,

$$\mathrm{var}_{\mathrm{post}} = (\sigma^2 + \mu^{*2})\left[\frac{\Phi(x^*)}{\Phi(x)} + \frac{1-\Phi(x^*)}{1-\Phi(x)}\right] - \mu^{*2}\left[\frac{\Phi(x^*)}{\Phi(x)} + \frac{1-\Phi(x^*)}{1-\Phi(x)}\right]^2$$
$$+ (\sigma^3\varphi'(x^*) - 2\sigma^2\mu^*\varphi(x^*))\left[\frac{1}{\Phi(x)} - \frac{1}{1-\Phi(x)}\right] - \sigma^4\varphi^2(x^*)\left[\frac{1}{\Phi(x)} - \frac{1}{1-\Phi(x)}\right]^2$$
$$- 2\sigma^2\mu^*\varphi(x^*)\left[\frac{1}{\Phi(x)} - \frac{1}{1-\Phi(x)}\right]\left[\frac{\Phi(x^*)}{\Phi(x)} + \frac{1-\Phi(x^*)}{1-\Phi(x)}\right], \quad (18)$$

where

$$\varphi'(x^*) = \frac{\partial}{\partial x^*}\varphi(x^*).$$

We now apply the criteria discussed before to estimate the parameter θ. As the ACC and ALC are of particular interest and guarantee their posterior coverages and lengths only on average with respect to the marginal distribution, we focus mainly on these criteria and compare with the worst outcome criterion (WOC) criterion.

4.1 *Average coverage criterion*

Combining Equations (5), (11) and (14), we find the minimum n that satisfies

$$\int_{-\infty}^{0} A\left\{\int_{a(x,n)}^{x'} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2\right]\mathrm{d}\theta\right\}$$
$$\times \exp\frac{1}{\beta}\left[\theta\sum_{i=1}^{k-1} x_{(i)} - (n-k-1)x_{(k)}\right]\mathrm{d}x$$
$$+ \int_{0}^{\infty} B\left\{\int_{x'}^{a(x,n)+\ell} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2\right]\mathrm{d}\theta\right\}$$
$$\times \exp\frac{1}{\beta}\left[(k-1)x_{(k+1)} - \sum_{i=k+2}^{n} x_{(i)}\right]\mathrm{d}x \geqslant 1-\alpha \quad (19)$$

where x' is the average value of $x_{(k)}$ and $x_{(k+1)}$ defined earlier in Equation (7) and

$$A = \frac{n!}{(n-k)!}\left(\frac{1}{2}\right)^n\left(\frac{1}{\beta}\right)^k\frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{(n-2k)^2\sigma^2}{2\beta^2} + \frac{\mu(n-2k)}{\beta}\right],$$
$$B = \frac{n!}{k!}\left(\frac{1}{2}\right)^n\left(\frac{1}{\beta}\right)^{n-k}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{(n-2k)^2\sigma^2}{2\beta^2} + \frac{\mu(n-2k)}{\beta}\right]. \quad (20)$$

4.2 *Average length criterion*

From Equations (6), (7), (11), and (14), the inequality of ALC is given by

$$A\sigma\sqrt{2\pi}\int_{-\infty}^{0} \ell'\Phi(x)\exp\left[\frac{1}{\beta}\left(\sum_{i=1}^{k-1} x_{(i)} - (n-k-1)x_{(k)}\right)\right]\mathrm{d}x$$
$$+ B\sigma\sqrt{2\pi}\int_{0}^{\infty} \ell'[1-\Phi(x)]\exp\left[\frac{1}{\beta}\left((k-1)x_{(k+1)} - \sum_{i=k+2}^{n} x_{(i)}\right)\right] \leqslant \ell, \quad (21)$$

where the length ℓ', corresponding to the HPD interval, is found for each given x and n by solving

$$\int_{a(x,n)}^{x'} \frac{[\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2 \mathrm{d}\theta\right] + \int_{x'}^{a(x,n)+\ell} \frac{[1-\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2 \mathrm{d}\theta\right] = 1-\alpha. \quad (22)$$

4.3 *Worst outcome criterion*

According to Equations (8) and (14), the WOC is satisfied by choosing the minimum n, where

$$\inf_{x\in X}\left\{\int_{a(x,n)}^{x'} \frac{[\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2 \mathrm{d}\theta\right] + \int_{x'}^{a(x,n)+\ell} \frac{[1-\Phi(x)]^{-1}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(\theta - \left(\mu + \frac{\sigma^2}{\beta}(n-2k)\right)\right)^2 \mathrm{d}\theta\right]\right\} \geqslant 1-\alpha \quad (23)$$

and by using the posterior variance, which is given in Equation (8), and differentiating with respect to x, we obtain

$$x = \mu, \quad \text{whenever} \quad n = 2k. \quad (24)$$

In other words, for any given sample size n, the length of the HPD interval is maximised when the number of data points falling in the negative region of the posterior distribution is the same as the number of data points falling in the positive region.

5. General strategy for sample size determination

Computer algorithms were devised to attain the sample sizes relatively quickly. Each algorithm is composed of several subalgorithms. A description of the main subalgorithms is given below, followed by an outline of the steps required for each criterion.

5.1 *Algorithms objectives*

All algorithms employ a bisectional search strategy to arrive at the final sample size, which stops when the criterion is satisfied for n but not for $n-1$. For each possible value of n, the relevant criterion was evaluated, and the next candidate was chosen depending on the result of the previous criterion.

5.2 *Finding lower and upper limits of highest posterior density intervals*

Although the previous paragraph indicates how integrals with known limits can be evaluated, another frequent problem was to find the particular limits corresponding to HPD intervals. As indicated in Section 2, the method of solution depends on the normal density as it is unimodal. It also depends on whether a and ℓ are both unknown, or whether ℓ is given. In the latter case, lower (a) and upper ($a+\ell$) limits for unimodal densities can be found by solving the equation

$$\psi(a|x,n,\mu,\sigma,\beta) - \psi(a+\ell|x,n,\mu,\sigma,\beta) = 0,$$

where $\psi(\ )$ is given by Equation (14) .The case of a and ℓ both unknown is two dimensional, but can be approached through a combination of techniques already mentioned. Good starting points for the search are helpful in reducing the computing time. For example, first approximations for a and ℓ can often be obtained from the symmetric credible set, for which a is simply the lower $\alpha/2$ percentile of the appropriate normal density, and $a + \ell$ is the upper $\alpha/2$ point.

5.3 *Algorithm for average coverage criterion*

For the ACC, μ, σ, β, α, and ℓ are fixed constants, and the coverage depends on x. The left-hand side inequality (19) must be calculated and compared with the desired average coverage $1 - \alpha$. For each value of x, the upper and lower limits, which depend only on a, as well as the resulting definite integral can be calculated as indicated earlier. The sum is compared with $1 - \alpha$, and the process continues until convergence.

5.4 *Algorithm for average length criterion*

For the ALC, μ, σ, β, α, and ℓ are again fixed constants, and the length of the HPD interval depends on x. The minimum n such that Equation (22) is satisfied is sought. The vector $f(x)$ is calculated by using Equation (11).The length of the HPD interval, represented by the vector $\ell'(x, n)$, is found by taking the difference between the upper and lower limits found as indicated above in the case that both a and ℓ are unknown .The resulting vectors can then be compared with $1 - \alpha$, and the search continuous until convergence.

5.5 *Algorithm for worst outcome criterion*

For the WOC, μ, σ, β, α, and ℓ are fixed, with no averaging required. The value of x is determined by condition (24), if we accept this conjecture. Thus, for each n, we need to only calculate the left-hand side of inequality (23), this integral being calculated as indicated earlier.

6. Numerical example

Examining the required sample size and taking into consideration the predescribed methods, using computer simulation for n, $1 \leq n \leq 500$, the ALC method was not an acceptable method to search for the sample size required. As for the other two methods, ACC is found to be the easier-to-compute criterion and WOC is preferred when one desires a more conservative sample size, which guarantees the length and posterior coverage over all anticipated data x. Different choices of μ, σ, and β were considered and Table 1 shows that ACC and WOC criteria produce considerable difference in estimates of sample size.

Though safety has improved to such an extent that mid-air collisions are now rare as discussed by Brooker [6], data on aircraft position errors were collected by Ternov and Akselsson [27] viewing the flight level $\theta = 33,000$ ft.

Table 1. Values of n_{ACC} and n_{WOC} for $\alpha \leqslant 0.10$.

x	ℓ	k	n_{ACC}	n_{WOC}
1	4	2	4	17
2	3	3	5	9
7	21	6	8	25
17	22	2	3	16
17	24	5	7	22
21	25	6	7	24

Table 2. Estimated values of n_{ACC} and n_{WOC}.

k	x	n_{ACC}	n_{WOC}
1	31,000	4	9
2	32,000	3	11
3	32,000	10	13
4	34,000	6	15
5	35,000	6	17
6	33,000	7	19

The following results, taking into account the maximum-likelihood estimates for the choices of μ, σ, and β, were obtained.

Again, Table 2 shows that ACC and WOC criteria produce considerable difference in estimates of sample size, and it is clear that $n_{ACC} < n_{WOC}$.

7. Conclusion

The three methods, suggested by Joseph *et al.* [13], differ substantially in calculating sample sizes in the case under study. It is obvious that ALC turned out to be a difficult criterion to find the optimal sample size required, whereas the decision as to which approach is better in selecting the appropriate sample size is maintained from the fact that, for $1 - \alpha \geqslant 0.90$, $n_{ACC} < n_{WOC}$. Ternov and Akselsson [27] applied a new proactive method for identifying hazards to an air traffic control unit in Malmoe, Sweden. These hazards are due to multiple reasons: winds, temperature, aircraft weight, and controller wishes to reduce flying distance; a number of changes in the flight plan may give rise to hazards, thus the 'dynamic scenario'. The Dutch National Aerospace Laboratory (NLR), as shown by Ruigrok and Hoekstra [22], has conducted extensive human-in-the-loop simulation experiments in NLR's Research Flight Simulator (RFS), focused on human factors evaluation of Free Flight. A question was born – 'how low can you go?'. In climb and descent phases of the flight, priority rules are used to indicate which of the aircraft involved in a conflict needs to maneuver before accidents happen. However, extraordinarily large position errors under consideration may result from a facilities' malfunction or breakdown or crew's blunders. For collision risk calculation and for the assurance of a sufficiently high level of safety in a large navigation system, it is always preferable to distinguish a poor but usual operation from the accidental through estimation of the proportion of accidental performances and the associated error size.

As in many traditional sample size problems, the practical use of the proposed approach will be as a decision support tool. We do not expect investigators to blindly trust the proposed solution. Rather, we envision that an investigator may be operating under budget and resource constraints that allow for a narrow range of sample size choices. The proposed methods can guide the choice within that range by informing the investigator about the likely payoffs and decision summaries.

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