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A Computational Finite Difference Treatment for PDEs Including the Mixed Derivative Term with High Accuracy on Curved Domains

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Abstract

In this paper, a finite difference treatment for Partial Differential Equations (PDEs) with the mixed derivative term is described. The method depends on using a simple first order PDE for a new dependent variable. The method deals with any problem formulated by a single elliptic PDE or by an elliptic system of two PDEs. Applying this approach to problems with curved boundaries and regular regions will decrease the number of unknowns in each equation and at the same time will increase the number of algebraic equations linearly and the accuracy quadratically. Moreover, the consistency of the finite difference representation of the system is achieved. Also, the derived system is of the same type as the original one. Two numerical applications are given. An efficient numerical algorithm is designed.

Keywords: Finite difference; Elliptic system; Curved boundaries; Mixed derivative.
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1 Introduction

The subject of PDEs holds an exciting and special position in mathematics. PDEs were not consciously created as a subject but emerged in the 18^{th} century as ordinary differential equations failed to describe the physical principles being studied, [2].

Unfortunately, only a limited number of PDEs have been solved analytically. That is why numerical techniques constitute a very important part in the field of solving PDEs.

Numerical methods for solving boundary value problems have developed rapidly. Knowledge of these methods is important both for engineers and scientists. Several numerical methods can be applied for solving boundary value problems such as finite difference method, FD, [6], finite elements method and so on, [7, 9].

The problem for solving elliptic PDE with a mixed derivative term is very important. The applications of this problem taken from meterology, upper atmospheric dynamics, and oceanography can be found in [5]. In this paper, we solve the elliptic boundary value problem for PDEs with the mixed derivative term in two dimensional space in a regular region (elliptic domain), the treatment is based on a local five-point approximation scheme, [6], and using a simple auxiliary first order PDE for a new dependent variable. The proposed method is used to avoid the use of the four corner points and accordingly decreases the number of unknowns as shown in § 2. It is known that, the finite difference approximation to the second derivative on curved boundaries can have a leading error of order h (the length of a square mesh side), [6], and consequently larger error in computations. Fortunately, the given approach will overcome this problem by using an extrapolation polynomial, [1]. Also, the consistency of the finite difference representation of the system is given in§ 2. § 3, the proof that the derived system is of elliptic type is given. Two numerical applications will be discussed in § 4. Furthermore, a numerical algorithm is designed and explained briefly in § 5. Also, the numerical results of the two problems are given in § 6. Finally, the conclusion is stated.

2 Problem formulation and basic equations

Consider an elliptic second order PDE with the mixed derivative term of the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = F(x, y),\ \forall x, y \varepsilon \Omega, \tag{2.1}$$

with Dirichlet boundary condition

$$u = G\, on\, \delta\Omega, \tag{2.2}$$

where Ω and $\delta\Omega$ are, respectively, the region and the boundary on which the function u is defined while a, b and c are constants with $b^2 < 4ac$, G is continuous on $\delta\Omega$

Consequently, the following system of linear equations is obtained by replacing each term in (2.1) by the corresponding central second order finite difference approximation, with a square mesh of side h.

$$4a(u_{i+1,j} + u_{i-1,j}) + 4c(u_{i,j+1} + u_{i,j-1}) + b(u_{i+1,j+1} - u_{i-1,j+1} + u_{i-1,j-1} - u_{i+1,j-1}) - 8(a+c)u_{i,j} = 4h^2 f_{i,j} \qquad (2.3)$$

where $i, j = 0, 1, \cdots, R$; R $=\lfloor$ maximum length of projections of Ω on the x-and y-axis /h $\rfloor$, equivalently, R is the number of mesh lines in either x- or y-directions. Obviously, the value of any $u_{i,j}$ can be determined when the eight variables $u_{i+k,j+l}$; $k, l\varepsilon\{-1, 0, 1\}$, where k and l don't equal zero together, are known. For curved domains, we attempt to avoid the use of the four corner points. So, (2.1) can be rewritten as:

$$au_{xx} + \frac{\partial}{\partial y}(bu_x + cu_y) = F(x, y)$$

Let w be a new variable defined as:

$$bu_x + cu_y = w(x, y)$$

which adds one supplementary first order PDE, and the following system is obtained.

$$\begin{aligned} au_{xx} + w_y &= F(x, y), \\ bu_x + cu_y &= w. \end{aligned} \qquad (2.4)$$

It is well-known that, the finite difference approximation to the second derivative can have a leading error of order h on curved and irregular boundaries, [6], and consequently larger error in computations. Thus the proposed method will overcome this problem by using an extrapolation polynomial, such as the Lagrange of order 2, [1], given by

$$P(x, y) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2, y) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1, y) + \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0, y). \qquad (2.5)$$

Note that: formula (2.5) can be applied along the vertical mesh line via varying y-coordinates instead of x-coordinates.

For example, the point (i,j) lies inside the region while the point (i+1,j) is outside the region as shown in Fig.1.

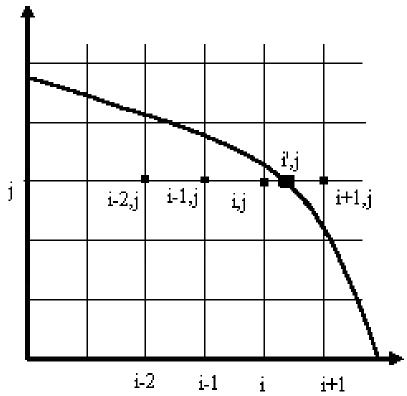

Fig.1.

To obtain the value of u at the point (i,j) along the mesh line j, substitute $x = ih$,$x_0 = i^{'} h$;$i < i^{'} < i + 1$, $x_1 = (i - 1)h$,$x_2 = (i - 2)h$, and f = u in (2.5), i.e., enforcing the extrapolating polynomial to satisfy the boundary condition. So, an approximate value of $u_{i,j}$ is found. Now, at each mesh point (i,j) there are two equations:

$$2a(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + h(w_{i,j+1} - w_{i,j-1}) = 2h^2 f_{i,j} \qquad (*)$$

and

$$b(u_{i+1,j} - u_{i-1,j}) + c(u_{i,j+1} - u_{i,j-1}) = 2hw_{i,j} \qquad (**)$$

So, for any n mesh points there exists a $2n \times 2n$ linear system of equations. Note that the number of mesh points $n = (R - 1)^2$.
The corresponding matrix of this linear system can have the following form

$$\text{C U = E},$$

where
$U = (u_{1,1}, w_{1,1}, u_{2,1}, w_{2,1}, \cdots , u_{R,1}, w_{R,1}, u_{1,2}, w_{1,2}, \cdots , u_{R,2}, w_{R,2}, \cdots ,$
$u_{1,R}, w_{1,R}, \cdots , u_{R-1,R}, w_{R-1,R})^T$,
$E = (-2au_{0,1}+hw_{1,0}+2h^2 f_{1,1}, bu_{0,1}+cu_{1,0}, hw_{2,0}+2h^2 f_{2,1}, cu_{2,0}, \cdots , -2au_{R,1}+$
$hw_{R-1,0}, -bu_{R,1} + cu_{R-1,0}, -2au_{0,2}, +2h^2 f_{1,2}, bu_{0,2}, 2h^2 f_{2,2}, 0, 2h^2 f_{3,2}, 0, \cdots ,$
$- 2au_{R,2} + 2h^2 f_{R-1,2}, -bu_{R,2}, -hw_{1,R} + 2h^2 f_{1,R-1}, bu_{0,R-1} - cu_{1,R}, -hw_{2,R} +$
$2h^2 f_{2,R-1}, -cu_{2,R}, \cdots , -2au_{R,R-1} - hw_{R-1,R} + 2h^2 f_{R-1,R-1}, -bu_{R,R-1} - cu_{R-1,R})^T$,

$$C = \begin{bmatrix} B & A & & & \\ -A & B & A & & \\ & & \cdot & & \\ & & -A & B & A \\ & & & -A & B \end{bmatrix}_{2n\times 2n}, B = \begin{bmatrix} B_1 & B_2 & & & \\ B_3 & B_1 & B_2 & & \\ & & \cdot & & \\ & & B_3 & B_1 & B_2 \\ & & & B_3 & B_1 \end{bmatrix}_{2(R-1)\times 2(R-1)},$$

$$A_{2(R-1)\times 2(R-1)} = diag\left\{A_1, A_1, \cdots, A_1\right\}, A_1 = \begin{bmatrix} 0 & h \\ c & 0 \end{bmatrix}, B_1 = \begin{bmatrix} -4a & 0 \\ 0 & -2h \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2a & 0 \\ b & 0 \end{bmatrix} and B_3 = \begin{bmatrix} 2a & 0 \\ -b & 0 \end{bmatrix}.$$

Now, we apply the above technique to an elliptic system of second order PDEs. Consider the system

$$Mu_{xx} + 2Nu_{xy} + Lu_{yy} = F(x,y), \tag{2.6}$$

where M, N, and L are 2×2 constant matrices, u and F are column functions. The biquadratic characteristic form of system (2.6) is

$$F(\eta_1, \eta_2) = det(M\eta_1^2 + 2N\eta_1\eta_2 + L\eta_2^2),$$

where $\eta_1 = \eta_1(x,y)$ and $\eta_2 = \eta_2(x,y)$ are the transformations used to get a simpler form of the system.

It is known that system (2.6)can be reduced to canonical form, with two independent parameters, [4]. In case of elliptic systems, the biquadratic characteristic form can be reduced, by a linear transformation of independent variables,into the following canonical form:

$$F(\eta_1, \eta_2) = (\eta_1^2 + \epsilon\eta_2^2)(\eta_1^2 + \tau^2\eta_2^2), \tag{2.7}$$

If $\epsilon = \tau = 0, F(\eta_1, \eta_2)$ has a quadruple real root. In case of$\epsilon = 1$ we have three cases

(I) $0 < \tau < 1, F(\eta_1, \eta_2)$ has two distinct pairs of complex roots

(II) $\tau = 1, F(\eta_1, \eta_2)$ has a pair of double complex roots

(III) $\tau = 0, F(\eta_1, \eta_2)$ has a pair of double complex roots and a double real root.

For example, consider the system

$$\begin{aligned} u_{xx} + (\frac{1}{2})v_{xy} + (\frac{1}{2})u_{yy} &= F_1(x,y), \\ (\frac{1}{2})v_{xx} + (\frac{1}{2})u_{xy} + v_{yy} &= F_2(x,y). \end{aligned} \tag{2.8}$$

for which

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, N = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix} and L = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the biquadratic characteristic form is

$$F(\eta_1, \eta_2) = det(M\eta_1^2 + 2N\eta_1\eta_2 + L\eta_2^2) = \begin{vmatrix} \eta_1^2 + 0.5\eta_2^2 & 0.5\eta_1\eta_2 \\ 0.5\eta_1\eta_2 & 0.5\eta_1^2 + \eta_2^2 \end{vmatrix}$$
$$= 0.5(\eta_1^4 + 2\eta_1^2\eta_2^2 + \eta_2^4) = 0.5(\eta_1^2 + \eta_2^2)^2,$$

i.e., the system (2.8) is an elliptic one. Rewrite system (2.8) as

$$\begin{aligned} u_{xx} + u_{yy} + w_y &= F_1(x, y), \\ v_{xx} + v_{yy} - w_x &= F_2(x, y). \end{aligned} \tag{2.9}$$

where $w = 0.5(v_x - u_y)$

In Equations (2.8) and (2.9), replacing each term by the corresponding finite difference formula, it is easy to see that in (2.8) each equation has nine entries. But in (2.9) each equation contains at most seven entries. So, we have $3n\times 3$n linear system of equations, which has the following form:

$$C^* U = E^*,$$

where E* is a column vector determined by the boundary values and U a column vector whose transpose is

$$U = (u_{1,1}, v_{1,1}, w_{1,1}, u_{2,1}, v_{2,1}, w_{2,1}, \cdots, u_{R,1}, v_{R,1}, w_{R,1}, u_{1,2}, v_{1,2}, w_{1,2}, \cdots, u_{R,2}, v_{R,2}, w_{R,2}, \cdots, u_{1,R}, v_{1,R}, w_{1,R}, \cdots, u_{R-1,R}, v_{R-1,R}, w_{R-1,R}),$$

$$C^* = \begin{bmatrix} B' & A' & & & \\ C' & B' & A' & & \\ & & \cdot & & \\ & & C' & B' & A' \\ & & & C' & B' \end{bmatrix}_{3n\times 3n}, B' = \begin{bmatrix} B_1' & B_2' & & & \\ B_3' & B_1' & B_2' & & \\ & & \cdot & & \\ & & B_3' & B_1' & B_2' \\ & & & B_3' & B_1' \end{bmatrix}_{3(R-1)\times 3(R-1)},$$

$$A'_{3(R-1)\times 3(R-1)} = diag\left\{A_1', A_1', \cdots, A_1'\right\}, C'_{3(R-1)\times 3(R-1)} = diag\left\{C_1', C_1', \cdots, C_1'\right\},$$

$$A_1' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -h \end{bmatrix}, B_1' = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 1 & -1 & -4h \end{bmatrix}, B_2' = \begin{bmatrix} 2 & 0 & h \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$B_3' = \begin{bmatrix} 2 & 0 & -h \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, and\, C_1' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & h \\ 0 & 0 & 0 \end{bmatrix}.$$

Consistency of the finite difference representation

To show the consistency of the finite difference, FD, representation of the system (2.4), let $U_{i,j}$ and $W_{i,j}$ be the exact solutions of $u_{i,j}$ and $w_{i,j}$, respectively, and $T_{i,j}$ and $T'_{i,j}$ be the truncation errors of $u_{i,j}$ and $w_{i,j}$,in the finite difference representation of (2.4)respectively. Apply Taylor's expansion [1], on the finite difference representation of (2.4) on the terms $U_{i+1,j}, U_{i-1,j}, W_{i,j+1}, \cdots$ we have

$$T_{i,j} = a(\frac{\partial^2 U}{\partial x^2})_{i,j} + (\frac{\partial W}{\partial y})_{i,j} + \frac{h^2}{3!}(\frac{\partial^3 W}{\partial y^3})_{i,j} - f_{i,j},$$

$T^{'}_{i,j} = b(\frac{\partial U}{\partial x})_{i,j} + c(\frac{\partial U}{\partial y})_{i,j} + \frac{h^2}{3!}(b\frac{\partial^3 U}{\partial x^3} + c\frac{\partial^3 U}{\partial y^3})_{i,j} - W_{i,j},$

Clearly, when h tends to zero, both $T_{i,j}$ and $T^{'}_{i,j}$ tend to zero for all i,j .

3 Ellipticity of the deduced system

Maintaining the ellipticity will be of advantage in the numerical treatment. To prove the ellipticity of the derived system, we will recall first the conditions that must be satisfied, [3].

Let Ω be a bounded domain in $\mathbf{R}^d$ and consider a system of n PDE's, written in the matrix form

$$L_\Omega u = F_\Omega, \tag{3.1}$$

where $L_\Omega = [L_{ij}]_{1\leq i,j\leq n}$ is a matrix of partial differential operators in which $L_{ij} = L_{ij}D = \sum_{|\alpha|\leq k} C_\alpha D^\alpha$. Here, the usual multi-index notation is being used. $D^\alpha = D_1^{\alpha_1}\cdots D_d^{\alpha_d}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Each L_{ij} can be considered as a polynomial in $D = \frac{\partial}{\partial x}$. We say that$L_{ij}$ is of order k, if there exists $c_\alpha \neq 0$ with $|\alpha| = k$.
Assume that there exist two n indices which are m_i and $m_i^{'}$, where $1 \leq i \leq n$ such that

$$order\, of\, L_{ij} \leq m_i + m_i^{'} \tag{3.2}$$

For given m_i ,$m_i^{'}$ define the principle term $L_{ij}^p(D) = \sum_{|\alpha|=m_i+m_i^{'}} C_\alpha D^\alpha$

Note that $L_{ij}^p(D)$ may vanish although $L_{ij} \neq 0$. The matrix with entries $L_{ij}^p(D)$ yields the principal part $L_\Omega^p(D)$ of $L_\Omega(D)$. Replacing D by $\beta = (\beta_1, \beta_2, \cdots, \beta_d)$ one obtains a matrix -valued polynomial $L^p(\beta)$ in β. If the coefficients of L_Ω^p depend on $x\varepsilon\mathbf{R}^d$, we write $L^p(x,\beta)$ instead of $L^p(\beta)$ The ellipticity is defined by means of $det(L^p(x,\beta))$, stated as follows.
Definition,([3], Hackbusch, 1985.
The differential operator L_Ω is called uniformly elliptic if there exist numbers $m_i, m_i^{'}$ satisfying (3.2)and with any constants $\nu, \mu > 0$,we have

$$\nu|\beta|^{2m} \leq |detL^p(x,\beta) \leq \mu|\beta|^{2m}, \qquad \forall x, \beta\epsilon\mathbf{R}^d$$

where $|\beta|^2 = \beta_1^2 + \cdots + \beta_d^2$ and $2m = \sum_{i=1}^n m_i + m_i^{'}$,
Now, it will be proved that the resulting system is uniformly elliptic. First we rewrite system (2.4) with the differential operator as follows:

$$aL_{11}u + L_{12}w = f,$$

$$bL_{21}u + cL_{22}w = 0,$$

where $L_{11} = \frac{\partial^2}{\partial x^2}$, $L_{12} = \frac{\partial}{\partial y}$, $L_{21} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $L_{22} = -1$.
Clearly, when taking $m_1 = m_1^{'} = 1$ and $m_2 = m_2^{'} = 0$, we have

$$order\,of\,L_{ij} \leq m_i + m_i^{'} \quad \forall i, j = 1, 2$$

and
$$detL^p(x, \beta) = \begin{vmatrix} a\beta_1^2 & \beta_2 \\ b\beta_1 + c\beta_2 & -1 \end{vmatrix}$$

Since $b^2 < 4ac$, we consequently have $|detL^p(x, \beta)| \leq ac(1 + 2\frac{\beta_1\beta_2}{|\beta^2|})|\beta|^2$.
As a result of $(\beta_1 - \beta_2)^2 \geq 0$, we get $|detL^p(x, \beta)| \leq 2ac|\beta|^2$. Also, with the same manner we can obtain $|detL^p(x, \beta)| \geq \frac{1}{2ac}(2 + \frac{\beta_1\beta_2}{|\beta^2|})|\beta|^2 \geq \frac{5}{4ac}|\beta|^2$.
Thus $\nu = \frac{5}{4ac}$ and $\mu = 2ac$. So the system is uniformly elliptic.

4 Numerical applications

Here, two numerical applications of the approach are given. The first application is to solve a single elliptic PDE(demonstrated in detail). The other is to solve a system of two PDEs (stated briefly), since it follows the same steps of the first one.

Application 1. Consider the elliptic PDE

$$u_{xx} + u_{xy} + u_{yy} = (\tfrac{\alpha^2+\beta^2}{2\alpha^2\beta} + \tfrac{\beta^4x^2+\alpha^2\beta^2xy+\alpha^4y^2}{4\alpha^4\beta^2})u(x, y),\ \forall x, y\epsilon\Omega,$$

.
with the boundary condition u=1 on $\delta\Omega$,
where$\delta\Omega$ is the curved boundary given by $\beta^2x^2 + \alpha^2y^2 = \alpha^2\beta^2$.
whose exact solution is $u(x, y) = e^{\frac{\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2}{4\alpha^2\beta}}$.
Clearly, the above system is similar to (2.4) with a=c=1 and also b=1. Hence, the system can be written as

$$\begin{aligned} u_{xx} + w_y &= (\frac{\alpha^2+\beta^2}{2\alpha^2\beta} + \frac{\beta^4x^2+\alpha^2\beta^2xy+\alpha^4y^2}{4\alpha^4\beta^2})u(x, y), \\ u_x + u_y - w &= 0. \end{aligned} \tag{4.1}$$

Using the corresponding finite difference formulae for the terms u_{xx}, u_x, u_y and w_y, then simplifying the result, one can easily obtain.
Let

$$K_{i,j} = (\frac{2(\alpha^2+\beta^2)}{\gamma} + \frac{4(\beta^4(ih)^2 + \alpha^2\beta^2ijh^2 + \alpha^4(jh)^2)}{\gamma^2}) \quad and \quad ,\gamma = 4\alpha^2\beta. \tag{4.2}$$

$$u_{i,j} = (u_{i+1,j} + u_{i-1,j} + \tfrac{h}{2}(w_{i,j+1} - w_{i,j-1}))/(2 + h^2K_{i,j}) \tag{4.2(a)}$$

$$w_{i,j} = (u_{i+1,j} - u_{i-1,j} + u_{i,j+1} - u_{i,j-1})/(2h) \tag{4.2(b)}$$

As a result of symmetry, the problem will be solved in the first quarter of the ellipse only, see[7]. More precisely,

$$u_{i,j} = u_{i,-j} = u_{-i,\pm j}, where\, i\epsilon\{0, 1, \cdots, \lfloor\frac{\alpha}{h}\rfloor\} and\, j\epsilon\{0, 1, \cdots, \lfloor\frac{\beta}{h}\rfloor\} \tag{4.3}$$

So, if one of the two points (i-1,j) or (i,j-1) lies outside the first quarter, i.e., for i=0 or j=0 the values of $u_{-1,j} and u_{i,-1}$ are determined directly from(4.3).

The system (4.2a,b) is applied if the points (i+1,j) and (i,j+1) are inside $\overline{\Omega} = (\Omega \bigcup \delta\Omega)$.While the situation is more complicated when one or both of these points lies outside $\overline{\Omega}$. If (i, j) doesn't lie outside $\overline{\Omega}$ and either (i+1,j) or (i,j+1)lies outside $\overline{\Omega}$, then different cases are generated which can not be solved by (4.2). So, we treat each of which separately in the following:

case 1.(i+1,j) is the only point existing outside $\overline{\Omega}$. $u_{i,j}$ should be determined by using the extrapolation formula (2.5) along the horizontal mesh line j. In this case, put $x_0 = x_{int} = i'h = (\alpha^2(1 - \frac{(jh)^2}{\beta^2}))^{\frac{1}{2}}$, $x_1 = (i-1)h, x_2 = (i-2)h$, and the value of u equals to one at the intersection point x_{int} of the mesh line j and $\delta\Omega$ as given in the boundary conditions, see Fig.1. Let

$$\theta_x = x_{int-ih} \tag{4.4}$$

Then $u_{i,j}$ is determined by

$$u_{i,j} = \frac{2h^2}{(x_{int}-(i-1)h)(x_{int}-(i-2)h)} + \frac{2(ih-x_{int})}{((i-1)h-x_{int})} u_{i-1,j} - \frac{(ih-x_{int})}{((i-2)h-x_{int})} u_{i-2,j} \tag{4.4a}$$

Unfortunately, we can't determine $w_{i,j}$ via using extrapolation formula, since w(x,y) is not known on $\delta\Omega$. Therefore, it can be found through the finite difference formula for u_x and the formula of u_y near a curved boundary when using a square mesh, [6]. More formally,

$$w_{i,j} = \frac{1-(1-\theta_x^2)u_{i,j} - \theta_x^2 u_{i-1,j}}{\theta_x(1+\theta_x)h^2} + \frac{u_{i,j+1} - u_{i,j-1}}{2h} \tag{4.4b}$$

Case 2.(i,j+1) is the only point existing outside $\overline{\Omega}$.Evidently, this case follows the same treatment as case 1 except that the extrapolation formula (2.5) will be applied along the vertical mesh line i and to the points $y_0 = y_{int} = (\beta^2(1 - \frac{(ih)^2}{\alpha^2}))^{\frac{1}{2}}, y_1 = (j-1)h, y_2 = (j-2)h.$
Let

$$\theta_y = y_{int-jh} \tag{4.5}$$

Then, we have

$$u_{i,j} = \frac{2h^2}{(y_{int}-(j-1)h)(y_{int}-(j-2)h)} + \frac{2(jh-y_{int})}{((j-1)h-y_{int})} u_{i,j-1} - \frac{(jh-y_{int})}{((j-2)h-y_{int})} u_{i,j-2} \tag{4.5a}$$

$$w_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{1-(1-\theta_y^2)u_{i,j} - \theta_y^2 u_{i,j-1}}{\theta_y(1+\theta_y)h^2} \tag{4.5b}$$

case 3. This case is more complicated , since both (i+1,j)and (i,j+1) lie outside $\overline{\Omega}$.If $i \geq j$ then $u_{i,j}$ will be determined using extrapolation formula (4.4(a)) along the mesh line j (To ensure the existence of the points (i-1,j) and (i-2,j) in $\overline{\Omega}$, see Fig.1), otherwise equation (4.5(a)) will be used.

While the calculation of $w_{i,j}$ depends on the formulae of u_x and u_y, since (i, j) exists near the curved boundary (as mentioned in Cases 1 and 2). Then

$$w_{i,j} = \frac{1-(1-\theta_x^2)u_{i,j} - \theta_x^2 u_{i-1,j}}{\theta_x(1+\theta_x)h^2} + \frac{1-(1-\theta_y^2)u_{i,j} - \theta_y^2 u_{i,j-1}}{\theta_y(1+\theta_y)h^2} \tag{4.6}$$

where θ_x and θ_y are given in the previous cases.

Finally, if a point (i, j) lies on the boundary,$u_{i,j}$ is determined from the boundary conditions but $w_{i,j}$ will be determined by using extrapolation formula (2.5) along the mesh line i or j depending on whether or not $i > j$ as explained for $u_{i,j}$ in Case 3. Then apply one of the following equations.

$$If \qquad (x,y)\epsilon\delta\Omega \tag{4.7}$$

Then

$$w_{i,j} = 3w_{i,j-1} - 3w_{i,j-2} + w_{i,j-3}, \qquad (along\, the\, mesh\, line\, i \quad , j > i), \tag{4.7a}$$

$$w_{i,j} = 3w_{i-1,j} - 3w_{i-2,j} + w_{i-3,j}, \qquad (along\, the\, mesh\, line\, j \quad , i \geq j), \tag{4.7b}$$

Application 2.

Consider the elliptic system:

$$\begin{aligned} u_{xx} + v_{xy} + 3u_{yy} &= F_1(x,y), \\ 3v_{xx} + 4u_{xy} + v_{yy} &= F_2(x,y). \end{aligned} \tag{4.8}$$

with the boundary condition u=v=1 on $\delta\Omega$, where $\delta\Omega$ is the curved boundary given by $\beta^2x^2 + \alpha^2y^2 = \alpha^2\beta^2$. Moreover,

$F_1(x,y) = (\frac{4\beta^4x^2+12\alpha^4y^2}{\gamma_1^2} + \frac{2\beta^2+6\alpha^2}{\gamma 1})e^{(\frac{\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2}{\gamma_1})} + \frac{4\alpha^2\beta^2xy}{\gamma_2^2}\, e^{(\frac{\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2}{\gamma_2})}$

$F_2(x,y) = (\frac{12\beta^4x^2+4\alpha^4y^2}{\gamma_2^2} + \frac{6\beta^2+2\alpha^2}{\gamma_2})e^{(\frac{\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2}{\gamma_2})} + \frac{16\alpha^2\beta^2xy}{\gamma_1^2}\, e^{(\frac{\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2}{\gamma_1})}$

where $\gamma_1 = 16\alpha^3\,\beta$ and $\gamma_2 = 4\alpha^2\,\beta^3$

Recall again that the approach is applied for this example takes the similar steps for the first one. So, we introduce the dependent variable w to derive the following system:

$$\begin{aligned} u_{xx} + u_{yy} + w_y &= F_1(x,y), \\ v_{xx} + v_{yy} + 2w_x &= F_2(x,y),. \\ v_x + 2u_y &= w(x,y). \end{aligned}$$

Clearly, we obtain the values of $u_{i,j}, v_{i,j}$, and $w_{i,j}, \quad \forall(i,j)\epsilon\overline{\Omega}$ according to the same classification of points in the first application (cases 1, 2 and

3). Furthermore, the equations of $v_{i,j}$ are derived similar to those for $u_{i,j}$.

The numerical solutions of this application are calculated via coding the following computational algorithm.

5 Computational algorithm

The two examples in the above section can not be solved without using a computer, so, a computational algorithm is designed. The algorithm technique is based on Gauss Seidel iteration method and the decision tree, which will be given in Fig.2.

The first step of the algorithm is to store information about the location of every mesh point (i, j) in $\overline{\Omega}$,since these information are needed repeatedly. This can be easily done via using 2-dimensional array, say loc, in which we store the following information: Let $\delta\Omega$ be an ellipse given by $\beta^2x^2 + \alpha^2y^2 = \alpha^2\beta^2$, where α and β are the lengths of the semi-major and semi-minor axes of the ellipse respectively. Then $\forall i\epsilon\{0, 1, \cdots, \lfloor\frac{\alpha}{h}\rfloor\}$, and $j\epsilon\{0, 1, \cdots, \lfloor\frac{\beta}{h}\rfloor\}$ where h is the side length of the square mesh. Put $: loc_{i,j} \leftarrow 1$ if $(\beta^2(ih)^2 + \alpha^2(jh)^2 < \alpha^2\beta^2, loc_{i,j} \leftarrow 2$ if equality holds, or $loc_{i,j} \leftarrow 3$ otherwise.

Next, follow the designed decision tree given in Fig. 2. The steps and the knowledge base of the algorithm are outlined:

1. Get loc for mesh point (i,j), for all i, j lie in the square region;
2. For all (i, j) in the square region do
3. If $loc_{i,j} = 3$ then nothing do;
4. If $loc_{i,j} = 2$ then $u_{i,j} \leftarrow u$ on $\delta\Omega$ and if $j > i$, calculate $w_{i,j}$ by (4.7(a)); otherwise by (4.7(b));
5. If $loc_{i,j} = 1$ and all surrounding 4 points exist in $\overline{\Omega}$,then calculate $u_{i,j}$ and $w_{i,j}$ by (4.2);
6. If $loc_{i,j} = 1\,\&\, loc_{i+1,j} = 3\,\&\, loc_{i,j+1} \neq 3$ then use (4.4) to get $u_{i,j}$ and $w_{i,j}$;
7. If $loc_{i,j} = 1\,\&\, loc_{i,j+1} \neq 3\,\&\, loc_{i+1,j} = 3$ then use (4.5) to get $u_{i,j}$ and $w_{i,j}$;
8. If $loc_{i,j} = 1\,\&\, loc_{i+1,j} = loc_{i,j+1} = 3$ then use (4.6) to calculate $w_{i,j}$ and if $i \geq j$, use (4.4(a)) to calculate $u_{i,j}$ and (4.5(a))otherwise;
9. If $loc_{i,j} = 1\,\&\,\{(i = 0) \bigvee (j = 0)\}$ then in the former case use (4.3)to calculate $u_{-1,j}$ and in the later case calculate $u_{i,-1}$ after that obtain $u_{i,j}$ and $w_{i,j}$ by (4.2);
10. Repeat steps 2 through 9 until $|(the\, old\, value - the\, new\, value)\, of\, u_{i,j}| < \varepsilon$ $\forall(i, j)$ of with $loc_{i,j} \leq 2$ $(\varepsilon \leq 10^{-6})$.

The second application that is given by (4.8) contains two functions : u and v. After applying the suggested approach, the resulting system (4.9) has three u, v and w. The above algorithm can solve this system with slight modifications. Obviously, the location of a point on the square region w.r.t. $\overline{\Omega}$ is the same as the first application. The main modification, from steps 2 to

9 of the above algorithm, is to use suitable equations for calculating $u_{i,j}, v_{i,j}$ and$w_{i,j}$ for all (i, j) $\epsilon\overline{\Omega}$. As the first application, we obtain the equations via applying finite difference formulae for $u_{xx}, u_{yy}, v_{xx}, v_{yy}, w_x$ and w_y. In special positions, we employ the extrapolation formulae and also the formulae of the first derivatives near the curved boundary when using a square mesh. The termination condition(step 10) in this case, will be modified to test $u_{i,j}$ and $v_{i,j}$ for all $(i,j)\epsilon\overline{\Omega}$ simultaneously. Then we guarantee that this version can be applied to solve the second application.
we used Pascal language to implement the two algorithms, and the results (for fixed α and β and various values of h) are given in the next section.

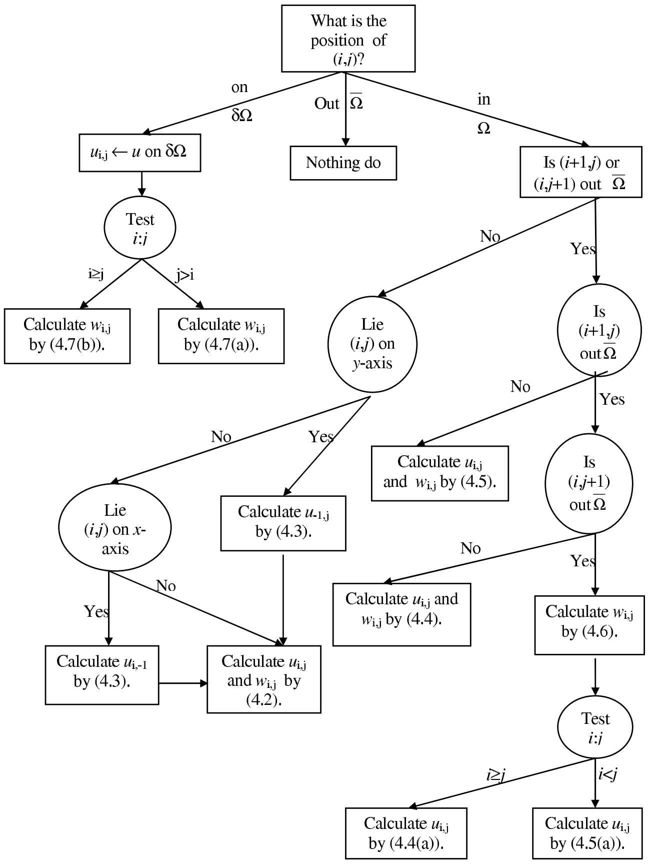

Fig.2: The decision tree of If-Then rules of the computational algorithm.

6 Numerical results

In this section, the computational results of applications one and two are given. The results are summarized both graphically and in tables. The computations have been performed for several values of h (0.05, 0.1 and 0.2) and for $\alpha = 2$ and $\beta = 1$. In the graphs, comparisons are made for several values of h with the exact solution at some fixed values of both x and y. While, in the tables the comparisons are made with the variation of both x and y.

Application 1.

Here, the exact solution is $u(x,y) = e^{\frac{\beta^2 x^2 + \alpha^2 y^2 - \alpha^2 \beta^2}{4\alpha^2 \beta}}$.
In Fig.3, comparison is made between the values of u(x, y) calculated at h=0.05 and h=0.1 with the exact values for y=0.6 and x varies from 0 to 0.5.

While In Fig.4, comparison is made between the values of u(x, y) calculated at h=0.05 and h=0.1 with the exact values for y=0.4 and x varies from

0 to 0.5.

Table 5 shows the values of u(x,y) with the variation of both x and y at h=0.1 and the exact values and the values of the error at each point.

Table 6 shows the values of u(x,y) with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

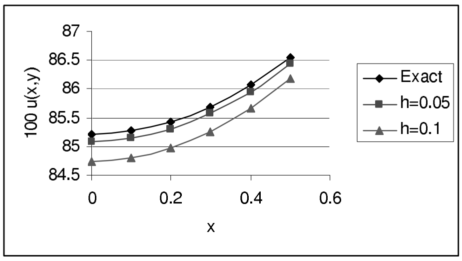

Fig.3: $0 \le x \le 0.5$, y=0.6, $u(x,y)\times 10^2$.

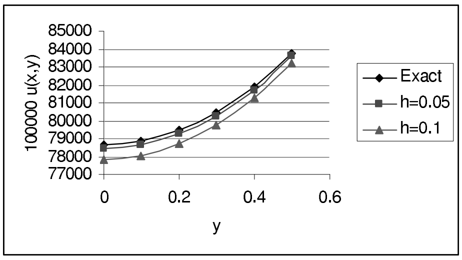

Fig.4: $0 \le y \le 0.5$, x=0.4, $u(x,y)\times 10^5$.

Application 2.

Here, the exact solution is $u(x,y) = e^{\frac{\beta^2 x^2 + \alpha^2 y^2 - \alpha^2 \beta^2}{16\alpha^2\beta}}$ and $v(x,y) = e^{\frac{\beta^2 x^2 + \alpha^2 y^2 - \alpha^2 \beta^2}{4\alpha^2\beta}}$

In Fig.7 comparison is made between the values of u(x, y) calculated at h=0.05 and h=0.1 with the exact values for y=0.2 and x varies from 0 to 0.8.

In Fig. 8, comparison is made between the values of v(x, y) calculated at h = 0.05 and h = 0.1 with the exact values at y = 0.2 and x various from 0 to 1.2.

In Fig. 9, comparison is made between the values of u(x, y) calculated at h = 0.1 and h = 0.2 with the exact values at x = 0.6 and y various from 0 to 0.8.

In Fig.10, comparison is made between the values of v(x, y) calculated at h = 0.1 and h = 0.2 with the exact values at x = 0.6 and y various from 0 to 0.8.

Table 11 shows the values of u(x,y) with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

Table 12 shows the values of v(x,y) with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

Table 5 shows the values of $u(x,y)$ with the variation of both x and y at h=0.1 and the exact values and the values of the error at each point.

x	y	Exact value	Approximation $h = 0.1$	Error $\times 10^3$
0.0	0.9	0.953610473132626	0.952597894084516	1.01258
0.4	0.9	0.963194417720822	0.962474648891825	0.719769
0.1	0.8	0.914502570801401	0.912212375048421	2.2902
0.5	0.8	0.928323507245164	0.926551317209248	1.77219
0.2	0.7	0.882496902584595	0.879221454425301	3.27545
0.9	0.7	0.926005597072012	0.924452294348387	1.5533
0.0	0.6	0.852143788966211	0.847311400111482	4.83239
1.1	0.6	0.919086534013519	0.917513257353823	1.57328
0.1	0.5	0.829547423332751	0.823749817730736	5.79761
0.8	0.5	0.862862438321375	0.859336157250918	3.52628
0.3	0.4	0.815156630128129	0.808779557259368	6.37707
0.9	0.4	0.852676545303328	0.848883669252707	3.79288
0.8	0.3	0.829029118180400	0.824109442500587	4.91968
1.8	0.3	0.975309912028333	0.975006350616978	0.303561
0.5	0.2	0.799015447388797	0.791974650918264	7.0408
1.0	0.2	0.837360999335754	0.833111451911724	4.24955
1.1	0.1	0.842084427143382	0.838126987151101	3.95744
1.5	0.1	0.898637995719454	0.896936932534139	1.70106
1.0	0.0	0.829029118180400	0.824314220433916	4.7149
1.9	0.0	0.975919671253259	0.975715080940785	0.20459

Table 5.

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Table 6 shows the values of $u(x,y)$ with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

x	y	Exact value	Approximation $h = 0.05$	Error $\times 10^3$
0.0	0.9	0.953610473132626	0.953284196809204	0.326276
0.4	0.9	0.963194417720822	0.962936168606856	0.258249
0.1	0.8	0.914502570801401	0.913848525578185	0.654045
0.5	0.8	0.928323507245164	0.927811893809870	0.511613
0.2	0.7	0.882496902584595	0.881535228482164	0.961674
0.9	0.7	0.926005597072012	0.925523175541555	0.482422
0.0	0.6	0.852143788966211	0.850812642846140	1.33115
1.1	0.6	0.919086534013519	0.918585086515610	0.501447
0.1	0.5	0.829547423332751	0.827936262742043	1.61116
0.8	0.5	0.862862438321375	0.861802326795439	1.06011
0.3	0.4	0.815156630128129	0.813381891692600	1.77474
0.9	0.4	0.852676545303328	0.851542936122984	1.13361
0.8	0.3	0.829029118180400	0.827592675050506	1.43644
1.8	0.3	0.975309912028333	0.975224832078526	0.0850799
0.5	0.2	0.799015447388797	0.797053183350722	1.96226
1.0	0.2	0.837360999335754	0.836103188519676	1.25781
1.1	0.1	0.842084427143382	0.840918905000620	1.16552
1.5	0.1	0.898637995719454	0.898109587804336	0.528408
1.0	0.0	0.829029118180400	0.827662703934517	1.36641
1.9	0.0	0.975919671253259	0.975847412623462	0.0722586

Table 6.

In Fig.7, comparison is made between the values of $u(x,y)$ calculated at $h = 0.05$ and 0.1 with the exact values at $y = 0.2$ and x varies from 0 to 0.8.

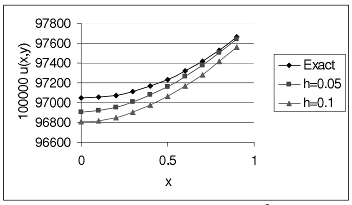

Fig.7: $0 \le x \le 0.8$, y =0.2, $u(x,y) \times 10^5$.

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In Fig.8, comparison is made between the values of $v(x,y)$ calculated at $h = 0.05$ and 0.1 with the exact values at y=0.2 and x varies from 0 to 1.2.

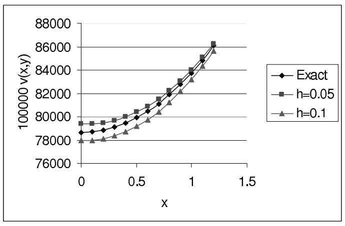

Fig.8: $0 \le x \le 1.2$ y =0.2, $v(x,y)\times 10^5$.

In Fig.9, comparison is made between the values of $u(x,y)$ calculated at h=0.1 and 0.2 with the exact values at x=0.6 and y varies from 0 to 0.8.

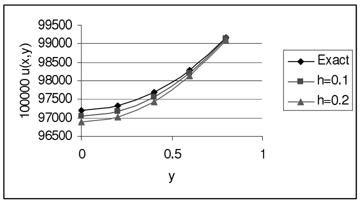

Fig.9: $0 \le y \le 0.8$, x =0.6, $u(x,y)\times 10^5$.

In Fig.10, comparison is made between the values of v(x,y) calculated at h=0.1 and 0.2 with the exact values at x=0.6 and y varies from 0 to 0.8.

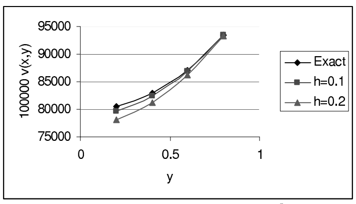

Fig.10: $0 \le y \le 0.8$, x =0.6, $v(x,y)\times 10^5$.

Table 11 shows the values of $u(x,y)$ with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

x	y	Exact value u (i,j)	Approximation $h = 0.05$	Error $\times 10^4$
0.2	0.8	0.989122096974591	0.988876652299946	2.45445
0.6	0.8	0.991597995801030	0.991553695434855	0.443004
0.4	0.6	0.981424687747777	0.980964063677216	4.60624
1.0	0.6	0.987886466783132	0.987868076163232	0.183906
0.8	0.4	0.978974190426360	0.978762022109536	2.12168
1.2	0.4	0.985111939603063	0.985109035945888	0.0290366
0.4	0.2	0.971659348942016	0.970764432953010	8.94916
1.4	0.2	0.985419835190368	0.985434571258056	0.147361
1.4	0.0	0.984188829934952	0.984199386406314	0.105565
1.8	0.0	0.994080092118170	0.994091623015671	0.115309

Table 11.

Table 12 shows the values of $v(x,y)$ with the variation of both x and y at h=0.05 and the exact values and the values of the error at each point.

x	y	Exact value $v(i,j)$	Approximation $h = 0.05$	Error $\times 10^2$
0.2	0.8	0.916218871650878	0.914141261404781	0.207761
0.6	0.8	0.934727720616028	0.936477473478094	0.174975
0.4	0.6	0.860707976425058	0.862474668287036	0.176669
1.0	0.6	0.907102341555802	0.908568705740346	0.146636
0.8	0.4	0.843664816596384	0.845667158286641	0.200234
1.2	0.4	0.886920436717158	0.888423936247137	0.15035
0.4	0.2	0.794533602503334	0.796770397495081	0.223679
1.4	0.2	0.889140511746448	0.890423478436096	0.128297
1.4	0.0	0.880293415834221	0.881693008414740	0.139959
1.8	0.0	0.953610473132626	0.954018757292296	0.0408284

Table 12.

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