On Numerical Counting of Prime, UPO, and the General Type of Posets According To Heights

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Abstract

This paper gives numerical values of non-isomorphic Prime, UPO, and the general types of posets with n elements ($n \leq 12$) according to heights. These countings depend on two aspects: First one is the modification of the algorithms in [1] and [6]. The second is to apply a technique which gives the general forms for the relations between the generating functions of two arguments, viz. the cardinality of a poset and its height, for non-isomorphic essentially series, essentially parallel, and $(+,\oplus)$-irreducible posets.

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Keywords

Counting, enumeration, poset, the height, prime, UPO, essentially, series, parallel, and $(+,\oplus)$-irreducible posets.

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§1 Introduction

All previous studies in the field of counting different classes of partially ordered sets (posets) treated the problem for getting exact or estimate numbers only according to the number of elements within the poset. Up to 1999, there are no counting researches for finding the numbers of non isomorphic n-element special type of posets regarding heights. Recently in [4] EL-Zahar and Khamis counted the number of non-isomorphic series-parallel n-element posets of height k, $1\leq k \leq n\leq 15$. While in this paper, we consider the problem of counting general form of posets according to their heights. Also, the author gives the numerical values of prime posets, and Uniquely Partially Orderable, UPO, posets on n elements and height k, where $1\leq k \leq n\leq 12$. The calculations depend on

1) The algorithm that creates the special connected type of posets via taking the height in account.
2) Applying suitable technique, if necessary to finding the numbers of total posets by using the results of the algorithm in (1).

In case of finding the numbers of the general posets, we use the technique that introduced by the author and El-Zhahar in [4]. This technique is developed to count the number of non-isomorphic n-element posets of height k in any class provided the numbers of non-isomorphic $(+,\oplus)$ -irreducible posets having cardinately n and height k. A brief explanation of this technique is given in §4. In fact, the technique is a modification of that derived by Stanely in [8] for obtaining the generating function of posets in terms of the generating of essentially series, essentially parallel, and $(+,\oplus)$-irreducible posets according to the number of elements within a poset only. The number of non-isomorphic $(+,\oplus)$-irreducible n-element posets of height k will be count via modifying the algorithms that introduced in [1] and [6]. The current algorithm is used to count the size of the class of connected posets regarding height. This is done through creating the precedence matrices in pseudo lexicographic order with rows appear in the bucket ordering. Then some tests are applied on matrices to choose those representing connected posets. For each accepted matrix, we will determine the height of the corresponding poset. The determination of height depends on finding the maximal chain within the poest. When we use this algorithm to find only $(+,\oplus)$-irreducible type, we add a test for excluding those matrices that represent essentially series posets. While in primality and UPO cases, we don't use the series-test but applying some other tests that check the primality and UPO characterizations, see [1]. The demonstration of the algorithm and its development are given in §3.

As a result of executing the algorithm, the numbers of non-isomorphic n element prime, connected UPO and (+,⨁)-irreducible posets of height k are obtained. To complete the calculation in the later case we use a simple method (see §5). This method shows how to apply the recurrence relations that are introduced in §4. Also, §5 includes 7 tables that present all numerical results of total posets, UPO posets and prime ones, each of which having up to 12 elements. It is clear that the running time is growing exponentially, so we stopped at n=12. A PC Pentium III 500 for 25 days is used to get the numerical values for (+,⨁) -irreducible posets. Also, in case of UPO and prime posets three PCs (PII-233, PII-350 and P III -500) for 26 days are used. For n=13, we expect that the execution time will take over 30 times of that for n=12. (The factor 30 is approximately the ratio between the number of total 12 element posets to 13 element posets). In [2], C. Claunier and N. Lygeros used nine Apollo workstations for six months to calculate the number of 13-element posets.

§2. Definitions

The terminologies, notation and basic concepts will follow that of [1], [6] and [7]. *A poset* P=(V,$<_p$) is a nonempty set V and an irreflexive, transitive and antisymmetric relation $<_p$ or < ,if P is understood, on V.

The *height* of a poset P is the maximum number of elements in a chain of P (a subset X of P is a chain if for every x,y∈X either x<y or y<x). The *dual* of P is the poset P^d= (V, $<^{-1}$). The *comparability graph* of P is denoted by *G(P)*. A graph G is *uniquely partially orderable* (*UPO)* if G=G(P) for some poset P and G=G(Q) implies Q=P or Q=P^d . We extended this definition by saying that P is UPO if G(P) has this characterization. For a poset P=(V,<) a subset U⊂V is called *P-autonomous* if ∀ x,y ∈ U and ∀ z∈V\U, x<z⇔ y<z and z<x ⇔z<y. A poset P is *prime* if it has no proper P-autonomous subset. It is well-known that P is UPO iff P has at most one non-trivial component and this component is either a 2-element chain or else every autonomous subset of which is an antichain. In fact prime posets are UPO. (For more information about prime and UPO posets and how to recognizing them, see [7]).

Let P' =(V' ,<') be a poset with element set V'= {v_1, ..., v_m} and let P_1=(V_1,$<_1$), ..., P_m = (V_m $<_m$) be posets with disjoint element sets. Then the *Lexicographic product* P'[P_1,..., P_m] of these posets is the poset P=(V,<) where V=V_1∪...∪ V_m and x<y iff either one of the following conditions is satisfied:

a) for some i, x,y ∈ V_i and x$<_i$ y , or
b) x∈ V_i and y ∈Vj and V_i<'V_j.

The poset P is called *decomposable* if P=P' [P_1, ... , P_m] where m> 1 and at least one of the P_i's is nontrivial, i.e., has more than one element. An indecomposable poset is called prime. If P' is a chain then P is the *ordinal sum* $\oplus$ of $P_1, P_2, \ldots$ and P_m and also it is called *essentially series* which is denoted by

$$P = P_1 \oplus P_2 \oplus \ldots \oplus P_m, \qquad m > 1.$$

Similar, when P' is antichain. Then P is called the *essentially parallel* or *disjoint sum (+)* of $P_1, P_2, \ldots P_m$ and it is denoted by

$$P = P_1 + P_2 + \ldots + P_m, \qquad m > 1.$$

Otherwise, P is called *(+,⊕)-irreducible* poset . In particular, If P is an (+,⊕)-irreducible having no proper P-autonomous it is also called prime poset.

To achieve the purpose of the paper, we want to count the number of non-isormphic n-element posets via knowing the smaller ones which are essentially series, essentially parallel, and (+,⊕)-irreducible ones. The two former types essentially series and essentially parallel are obtained by applying the technique of generating functions, that introduces in §4. Unfortunately, the last case may be difficult to obtain mathematically. Therefore, this case will be treated using an algorithmic approach. That is to generate, in a pseudo lexicographic order, precedence matrices with some special properties for representing posets. These matrices have the following properties.

Suppose that the poset P=(V,<) has a set of n elements V = $\{v_1, v_2, \ldots, v_n\}$. The *precedence matrix* of P is the nxn matrix A where A[i,j] = 1 whenever $v_i < v_j$ and 0 otherwise. Let R[i] and C[i], i=1, ... , n, denote respectively the row and column sums of A. Obviously R[i] and C[i] are the cardinalities of the successor and predecessor sets of v_i. In what follows we shall assume that the elements of P are ordered such that:

I) R[1]≥ R[2]≥ ... ≥R[n].
II) If R[i] = R[i+1] then C[i] ≤C[i+1].

Satisfying properties (I) and (II) lead to creating nonredandant precedence matrices that are upper triangular with rows appear in the Bucket sort.

Assume that A(n) denote the set of all precedence matrices, of n element posets, satisfying (I) and (II) above. Then a *pseudo-lexicographic ordering* on A(n) is defined as follows: For A,B ∈A(n), A proceeds B in pseudo lexicographic order if there is an (i,j) such that :

A[k,l] = B[k,l] ∀ k>i and for k=i and all $l<j$, whereas
A[i,j] = 0, and B[i,j] = 1.

Note that the first and last matrices in this order are, respectively, the precedence matrices of an antichain and a chain.

In the next section, we demonstrate briefly the algorithm, which generates the precedence matrices that represent, connected posets in pseudo lexicographic order. It is also used to count connected UPO, prime and (+,⊕) – irreducible posets according to heights. The algorithm is a modification of those algorithms that reported in [1] and [6].

§3. Counting Of Prime, UPO, and (+,⊕)-irreducible Posets

The algorithm consists of four major steps. Namely,

1) Generating the next precedence matrix of order n in the pseudo lexicographic order satisfying transitivity constraint, properties (I) and (II) in §2.
2) Checking whether or not the current matrix represents a connected posets, i.e. whose diagram representation is connected.
3) Calculating the height of the poset that is represented by the accepted matrix.
4) Determinating the weight of the matrix.

To obtain the number of non-isomorphic connected n-element posets we apply the above last three steps repeatedly on each accepted element of *A(n)* which is created by step (1) in the pseudo-lexicographic order. For all possible height k, the weights of accepted matrices are summed up to give, at the end of execution, the number of the specified type of height k. Neglecting the height, one can sum these numbers for all k to get the total number of connected posets with n element.

§§3.1. Generating the Next Matrix

To obtain the next matrix of a current matrix A in A(n), we use the following algorithm which is developed in [1].

Step 1 : Search the rows of A starting at the first row till the (n-l)st one. Each row i is searched from the nth entry and moving left until the (i+l)st entry. The search continues until the first zero is encountered. If no such zero is met then the matrix is the last one. Otherwise let A[pivot-row,pivot-column] is the first encountered zero element.

Step 2 : Put A[pivot-row, pivot-column] =1 and A[pivot-row,j] =0 for all j>pivot-column.

Step 3 : Adjust the transitivity of the matrix by putting A[pivot-row,k] = 1 whenever A[pivot-row ,j]= A[j,k] =1 for pivot row<j<k.

Step 4 : If now R[pivot-row] <R[pivot-row+l] then, starting at the nth entry of the pivot-row and moving left replace 0's by 1's until R[pivot-row] = R[pivot-row+ 1]).

Step 5 : For each $1 \leq i <$ pivot-row put A[i, j]=0 for each i<j≤n-R[pivot-row].

Step 6 : If for some l≤i<n, we have R[i]=R[i+ 1] and C[i]>C[i+ 1] then the matrix is not accepted. In this case goto *step 1*. Otherwise the next matrix is obtained and hence it is accepted.

§§3.2. Suggesting Tests

§§3.2.1. Connectivity Testing

Consider a precedence matrix A is obtain as a result of the algorithm in §§3.1. Here, we check whether or not A represents a connected poset.

Step 1 : Put TestedElements = ∅.
Step 2 : Put IncludedElements = {n} ∪ {j : A[j,n]=l}.
Step 3 : IF TestedElements = IncludedElements then goto *step 6*.
Step 4 : Put TestedElements = IncludedElements.
Step 5 : For each element "i" in TestedElements which is not used in any previous step; Put IncludedElements = TestedElements u{i}u{j : A[j,i]=l}∪ {k : A[i,k]=l} and then goto *step 3*.
Step 6 : If the Cardinality of IncludedElements = n then A is a one required matrix since it represents a connected poset, otherwise A is discarded because it represents a disconnected poset.

Any accepted matrix of the above algorithm represents a connected poset. It is well-known that not all connected posets are (+,⊕)-irreducible ones, then to obtain the class of (+,⊕)-irreducible posets, we must exclude those matrices that represent essentially series posets. In fact those excluded matrices are not represented matrices (+,⊕)-irreducible posets. This is done via the following algorithm which checks whether the matrix represents series posets or not.

§§3.2.2. The Seriestesting

This algorithm works similar to the connectivity testing. The procedure follows the next steps.

Step 1 : Put TestedElements = ∅.
Step 2 : Put IncludedElements = {n} ∪({l...n}\ {j : A[j,n]=l}).
Step 3 : If TestedElements = IncludedElements then goto *step 6*.
Step 4 : Put TestedElements = IncludedElements.
Step 5 : For each element i in TestedElements which is not used in any previous step; Put IncludedElements = TestedElements ∪ ({l...n}\{j : A[j,i]=1}) goto *step 3*.
Step 6 : If the Cardinality of IncludedElements = n then such matrix represents a series poset and will be discarded, otherwise the matrix is accepted.

Some essentially series posets are also UPO, so, in case of counting UPO the algorithm in §§3.2.2 cannot be applied. Since not all connected posets are UPO, then the following algorithm will be applied.

§§3.2.3. The UPO and Primality testing

As mentioned in [1], the UPO testing depends on obtaining the implication classes of a given poset P. Then if there exists only one implication class containing all pairs $(v_i, v_j) \in$ P(A[i,j]= l), then P is a UPO. Now, we outline how to find an implication class via using a matrix A.

Step 1 : Pick an arbitrary entry (i,j) in A such that A[i,j] = 1.
Step 2 : Put Mark on the entry (i,j).
Step 3: Mark every entry (k,l) which relates with **(i,j)** by the reflexive-transitive coluser of a force relation Γ , (for any two entries **(i,j)** and (p,q) such that A[i,j] =A[p,q]=l where $\mathbf{i} \leq \mathbf{p}$ and $\mathbf{j} \leq$ q, we have **(i,j)** Γ (p,q) iff either (i=p and A[j,q]=0)or (j=q and A[i,p]=0), see the definition of force relation in ([7] pp.43)).
Step 4 : If all entries of A are marked then a poset P is a UPO and goto step 5.
Otherwise A is neglected and go to find the next matrix in A(n) (§§3.1). .
Step 5 : Primality characterization.
Check whether a poset having only trivial antichains. This is done simplicity as follows: Search for a pair (i,j) s.t. for all k≤n, we have A[i,k] = A[j,k] and A[k,i]=A[kj].
If there exists such (i,j) , then a matrix represents UPO poset only otherwise it represents prime poset as well as it is UPO.
(The explanation of another method for primality testing is given in ([6],§3.2)).

Note that, the algorithm counts only the number of connected UPO poset on n elements and height k. But it is required to find the number of total ones which having at most one non-trivial component. This can be easily done by adding the number of connected n-element UPO having height k to the total number of (n-1) element UPO having height k. So, In both cases, either prime or only UPO we must determine the height of a poset. This is explained in the following subsection.

§§3J. Calculating the height of a Poset

Before finding the weight of an accepted matrix that represents either prime, UPO, or (+,⊕) -irreducible posets, we determine the height of a poset. This is done via counting the number of elements in the maximum chain of a poset. To achieving this, we assign a label for each element v in a poset P. The label L_v of v in P is the number of elements within the maximum predecessor chain of v in P. Then the height of P is the largest value of labels of the maximum elements of P. Now, we apply the following steps to find height of P, which is represented by the matrix A.
Step 1: Put L_i =l for all i having C[i]=0.

Step 2: Put L_k = 2 for all k,i satisfying that A[i,k]=l where i having C[i] =0 and ∄**j** s.t. A[i,j]= A[j,k]=l.

Step 3: Search for unlabeled element k whose all immediate predecessors having labeled. This is done by checking wether all element "i", satisfying that A[i,k]=l and ∄j s.t. A[i,j]=A[j,k] =1, are labeled or not.

Step 4: If no such k goto next step otherwise put L_k =L+l ; where L =max {L_i : for all i which are the immediate predecessors of k } and then goto *step3.*

Step 5: The height of P = max { L_k : for all k having R[k]=0}.

§§3.4. Counting the weight of a matrix

In [6] the author explained in details the computation of the weight of a poset by a fraction. So, here we give only a summary of the method.

First of all, the elements of posets are partitioned into blocks such that v_i,v_j belong to the same block iff R[i] = R[j] and C[i] = C[j].

Consider ∑ denotes the set of all permutations σ of {1,2,...,n} such that v_i and $v_{\sigma(i)}$ are in the same block for every 1≤ i ≤n. For $\sigma \in \sum$ denote by ∑ (A) the matrix obtained from A by applying σ to its rows and columns. All matrices σ(A), $\sigma \in \sum$ represent the isomorphic posets. We wish to count the number of these isomorphic copies. This number which is the cardinality of ∑, can easily determine by $r_1!\, r_2! \ldots r_k!$ where $r_1, r_2, \ldots, r_k$ are the sizes of blocks in A. There exist some of σ (A) is identical of A. To obtain the number of this σ, say M(A), which leading to σ (A) =A, we generate all possible permutations $\sigma \in \sum$. Then counting those permutations when applying them on the rows and columns of the matrix(A) produces the same matrix.

Finally the matrix A is assigned the weight M(A)/|∑|. The number of required posets having a certain height is the sum of the weights of the accepted matrices having the same heights. Note that the weight of the matrix having n blocks (the order of the matrix) is one in this case we don't calculate the number M(A) and this is aid to reduced the running time of an algorithm code.

The result of using algorithms §3.1, §§3.2.1, §§3.2.3, §3.3 and §3.4 is the numbers of the connected n-element UPO and n-element prime posets having height k, for any n,k≥1. In counting the total numbers of posets having height k, we first execute the algorithms in §3.1, §§3.2.1, §§3.2.2, §3.3 and §3.4 to enumerate the numbers of (+,⊕)-irreducible ones. We apply the following technique to have the required numbers.

§4. A Technique of Counting Number of Poset

Since to the calculations of this paper based directly on the technique derived in [4], we give the brief explanation of this technique. Also, we recall and will use the same notation, the same generating functions and the same classifications of subclasses of posets. Let ε be the class of non-isomorphic posets, which is closed under the series and parallel operations. Consider f_{nk} the number Of non-isomorphic n-element posets Of height k in ε. AlSO, denote to v_{nk}, u_{nk} and i_{nk} the number of non-isomorphic essentially series, essentially parallel,(+,⊕)-irreducible n-element posets having height k, respectively in the class ε. Obviously for $n \geq k \geq 1$, we have the following well-known relation

$$f_{nk} = v_{nk} + u_{nk} + i_{nk} \qquad (4.1).$$

consider the generating functions F(x,y), V(x,y), U(x,y) and I(x,y) which are defined respectively by the above numbers f_{nk}, v_{nk}, u_{nk} *and* i_{nk} respectively, as follows

$$F(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{nk}x^n y^k = \sum_{k=1}^{\infty} F_k(x)y^k,$$

$$V(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} v_{nk}x^n y^k = \sum_{k=1}^{\infty} V_k(x)y^k,$$

$$U(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} u_{nk}x^n y^k = \sum_{k=1}^{\infty} U_k(x)y^k, and$$

$$I(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} i_{nk}x^n y^k = \sum_{k=1}^{\infty} I_k(x)y^k \quad .$$

From equation (4.1), we have

$$F(x,y) = U(x,y)+ V(x,y)+ I(x,y) \qquad (4.2).$$

Consequently, from Lemma 3.1 in [4] we obtain the relation between generating functions V(x,y) and the others which is given by :-

$$V(x,y) =(U(x,y)+I(x,y))F(x,y) \qquad (4.3).$$

To obtain the required numbers which are the coefficients of F(x,y), we first consider the class $\mathcal{W}_k$. This class is denoted the class of posets in $\mathcal{E}$ with the property that each components of which has height k. Let $\mathcal{W}_{nk}$ be the number of non-isomorphic n-element posets in $\mathcal{W}_k$. Consider the generating function

$$W(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} w_{nk}\, x^n y^k = \sum_{k=1}^{\infty} w_k(x)y^k \quad .$$

The coefficients of W_k(x) which are the numbers $w_{nk}'s$ are determined by using the next relation that derived in [4] (see lemma 3.2). This relation calculates W_k(x) in terms of V_k(x) and I_k(x). The derived of the following relation is based on the Riddle's theorem which is used to getting the number of total graphs from smaller connected ones (see [5] p. 90).

$$1+W_k\,(x) = \exp\left(\sum_{i=1}^{\infty}\left(v_k\left(x^i\right) + I_k\left(x^i\right)\right) /i\right) \quad (4.4).$$

Now, We calculate F_k (x) in terms of W_k (x). This is given by the following equation, which is proved in Lemma (4.3) in [4].

$$F_k(x) = \begin{cases} W_k\,(x) & if\ k=1 \\ (1+\sum_{j=1}^{k-1} F_j(x))W_k(x) & if\ k \geq 2 \end{cases} \quad (4.5).$$

To obtain the coefficients of the generating functions F(x,y), V(x,y), and U(x,y), we must previously known the function I(x,y). The numbers i_{nk} for some special classes can be counted via using certain algorithm. In the above section, we demonstrate briefly one algorithm that is use to count (+,⊕)-irreducible posets for total posets.

§5.Numerical Results

The aim of this part is introduce a simple algorithm for treating the sophisticated recurrence relations of the technique introduced in §4. Then the steps of algorithm are developed to enable anyone to using.

Now to obtain the numerical results for the coefficients of U(x,y),V(x,y) and F(x,y) for certain k and n, apply the following steps.

Step 1 : Initial step

Put j =l; I_1 (x) =x; V_1 (x) = 0 ; U_1 (x) =$x^2 + x^3$+...(all n-element antichains); F_1 (x) = I_1 (x) + U_1 (x).

Step 2 : Recursive steps

j = j+ 1.

Step 3 : Get $J_j \subseteq \mathcal{E}$ which consists of all non -isomorphically (+,⊕)-irreducible posets of exact length j.

Then i_{mj} = the number of m-element of J_j ;(using the algorithm that is given in §3).

Step 4 : Find V_j(x) using equation 4.3 which is expended by

$$V_j\,(X) = \sum_{i=1}^{j} F_{j-i}(x)(\,U_i\,(x) + I_i\,(x)).$$

Step 5 : Using equation(4 .4) to Obtain W_K (x). This is done as follows :

Put B_j (X) = $\sum b_{nj}x^n = \sum_{j=1}(V_j(x^i) + I_j(x^i)) / i$.

Then pb_{pj} =$\sum_{d/p} d(v_{dj} + i_{dj})$. (the summation takes over all divisors of p)

Consequently, for all possible m, we have

$$mw_{mj} = mb_{mj} + \sum_{d=1}^{m-1} db_{dj}w_{(m-d)j}.$$

Step 6 : Determine F_j(x) using equation (4.5).

- *Step 7 :* Using equation (4.2) to obtain U_j (x) = F_j (x)- V_j (x)-I j(x) .

Step 8 : Repeat steps from 2 to 7 until to find the required F_k (x).

The complete numerical results for the coefficients of I(x,y) (rsulting from the algorithm that is given in §3), V(x,y), U(x,y) and F(x,y) (resulting from the above simple algorithm) for k,n ≤ 12 are appered in tables (1),(2), (3) and (4) respectively. Tables (5) and (6) consist of the algorithmic results of the numbers of non- isomorphic n-element prime and connected UPO of height k, where $1\leq k \leq n \leq 12$,respectively . Finally table (7) includes the calculated values of the numbers of non-isomorphic UPO with the same limits.

Remark

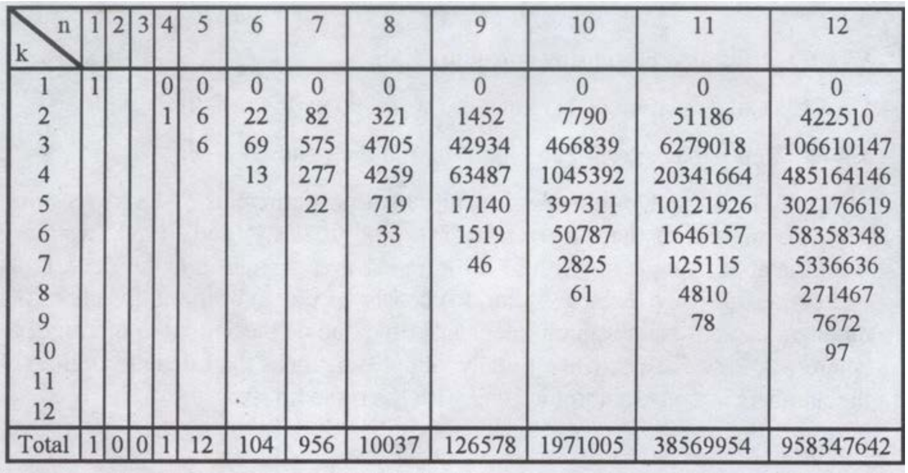
The results of this paper agree with that introduced in [1] in both cases of UPO and prime posets for up to eleven elements regardless height. The author detected that there exist some errors in the case of UPO. These errors are in fact misprinting errors in the numbers of UPO at n=9 and 10. The correct numbers are presented in table (7) at total row and columns 9 and 10. In addition in tables 5 -7 we give new calculated results at n= 12. Also, the counting of general posets agrees with that numbers given in [7,2,3] up n=9,11, and 12 without regarding height, respectively. While the number of total posets at n= 10 is not coincide with the value that is given in [7].

Note That :-

In the following tables, all missing numbers are zeroes.

Table (1)

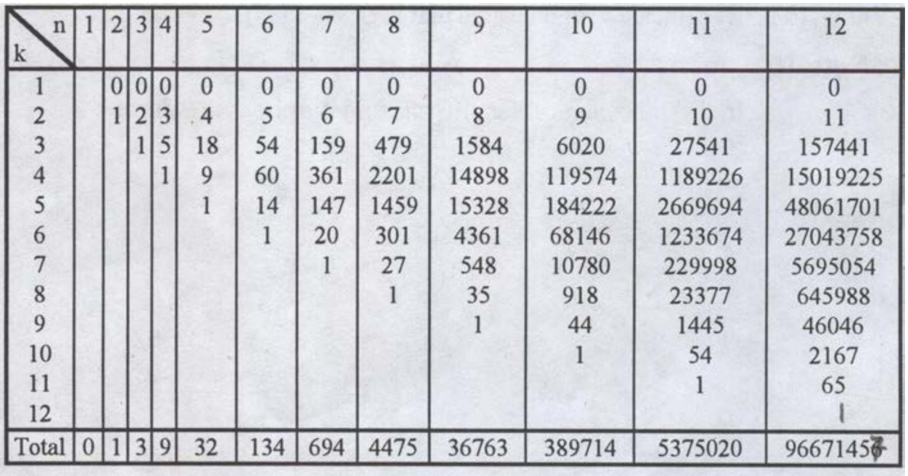
The numbers of non-isomorphic (+,⊕) -irreducible n-element posets of height k.



n k	1	2	3	4	5	6	7	8	9	10	11	12
1	1			0	0	0	0	0	0	0	0	0
2				1	6	22	82	321	1452	7790	51186	422510
3					6	69	575	4705	42934	466839	6279018	106610147
4						13	277	4259	63487	1045392	20341664	485164146
5							22	719	17140	397311	10121926	302176619
6								33	1519	50787	1646157	58358348
7									46	2825	125115	5336636
8										61	4810	271467
9											78	7672
10												97
11												
12												
Total	1	0	0	1	12	104	956	10037	126578	1971005	38569954	958347642

Table (2)

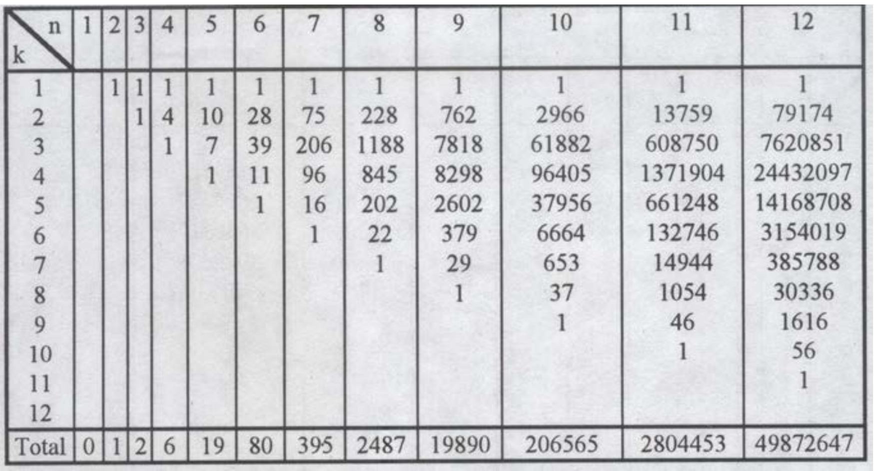
The numbers of non-isomorphic essentially series n-element posets of height k.



n k	1	2	3	4	5	6	7	8	9	10	11	12
1		0	0	0	0	0	0	0	0	0	0	0
2		1	2	3	4	5	6	7	8	9	10	11
3			1	5	18	54	159	479	1584	6020	27541	157441
4				1	9	60	361	2201	14898	119574	1189226	15019225
5					1	14	147	1459	15328	184222	2669694	48061701
6						1	20	301	4361	68146	1233674	27043758
7							1	27	548	10780	229998	5695054
8								1	35	918	23377	645988
9									1	44	1445	46046
10										1	54	2167
11											1	65
12												1
Total	0	1	3	9	32	134	694	4475	36763	389714	5375020	9667145~~6~~7

Table (3)

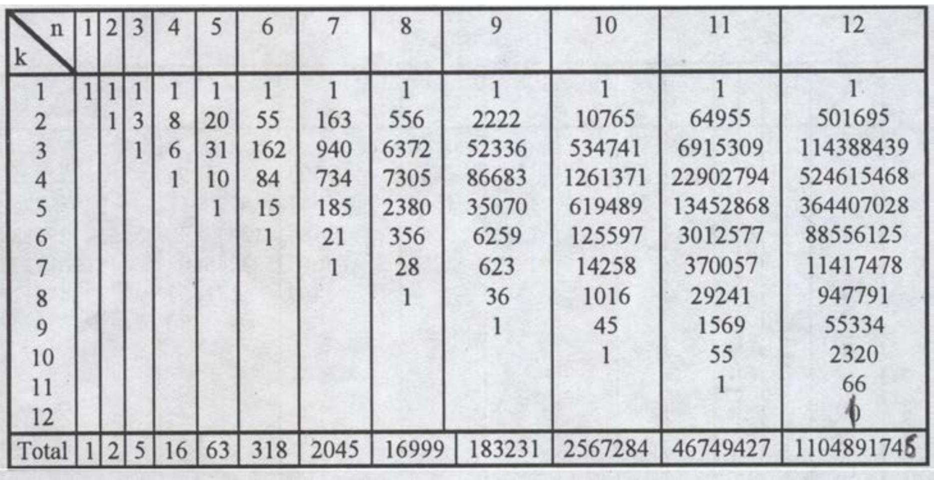
The numbers of non-isomorphic essentially parallel n-element posets of height k.



n / k	1	2	3	4	5	6	7	8	9	10	11	12
1		1	1	1	1	1	1	1	1	1	1	1
2			1	4	10	28	75	228	762	2966	13759	79174
3				1	7	39	206	1188	7818	61882	608750	7620851
4					1	11	96	845	8298	96405	1371904	24432097
5						1	16	202	2602	37956	661248	14168708
6							1	22	379	6664	132746	3154019
7								1	29	653	14944	385788
8									1	37	1054	30336
9										1	46	1616
10											1	56
11												1
12												
Total	0	1	2	6	19	80	395	2487	19890	206565	2804453	49872647

Table (4)

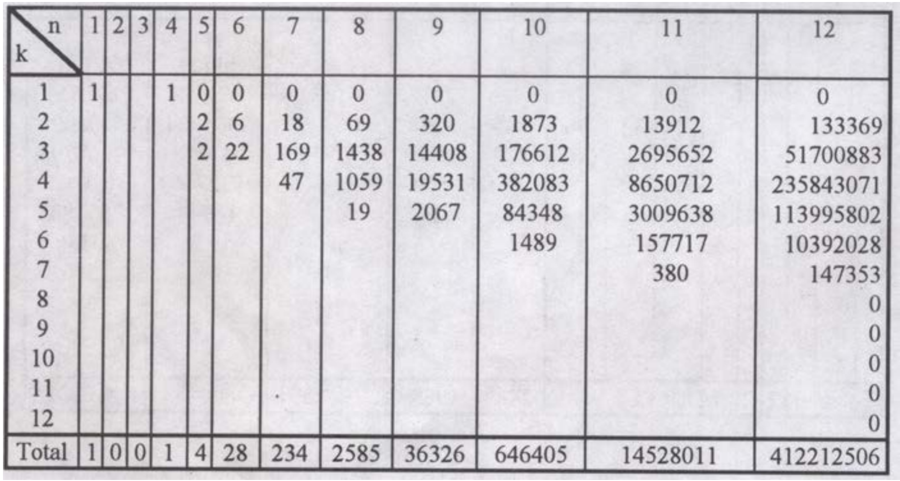
The numbers of total non-isomorphic n-element posets of height k.



n / k	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	8	20	55	163	556	2222	10765	64955	501695
3			1	6	31	162	940	6372	52336	534741	6915309	114388439
4				1	10	84	734	7305	86683	1261371	22902794	524615468
5					1	15	185	2380	35070	619489	13452868	364407028
6						1	21	356	6259	125597	3012577	88556125
7							1	28	623	14258	370057	11417478
8								1	36	1016	29241	947791
9									1	45	1569	55334
10										1	55	2320
11											1	66
12												~~0~~ 1
Total	1	2	5	16	63	318	2045	16999	183231	2567284	46749427	110489174~~5~~6

Table (5)

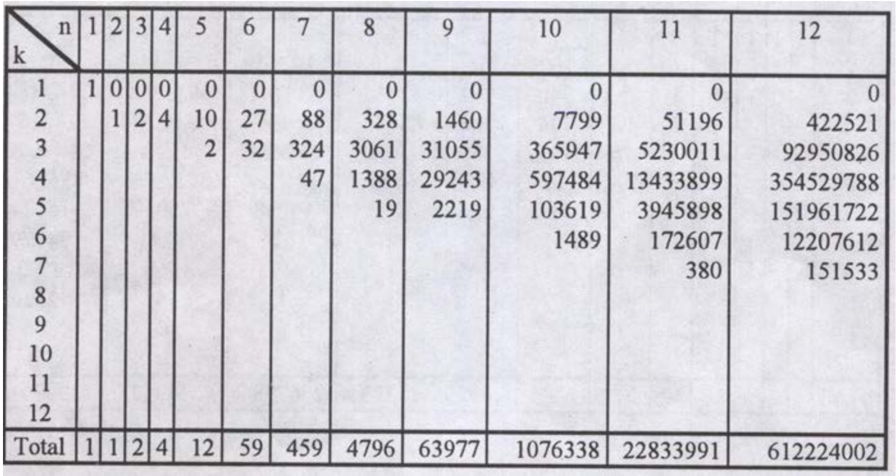
The numbers of non-isomorphic prime n-element posets of height k.



k \ n	1	2	3	4	5	6	7	8	9	10	11	12
1	1			1	0	0	0	0	0	0	0	0
2					2	6	18	69	320	1873	13912	133369
3					2	22	169	1438	14408	176612	2695652	51700883
4							47	1059	19531	382083	8650712	235843071
5								19	2067	84348	3009638	113995802
6										1489	157717	10392028
7											380	147353
8												0
9												0
10												0
11												0
12												0
Total	1	0	0	1	4	28	234	2585	36326	646405	14528011	412212506

Table (6)

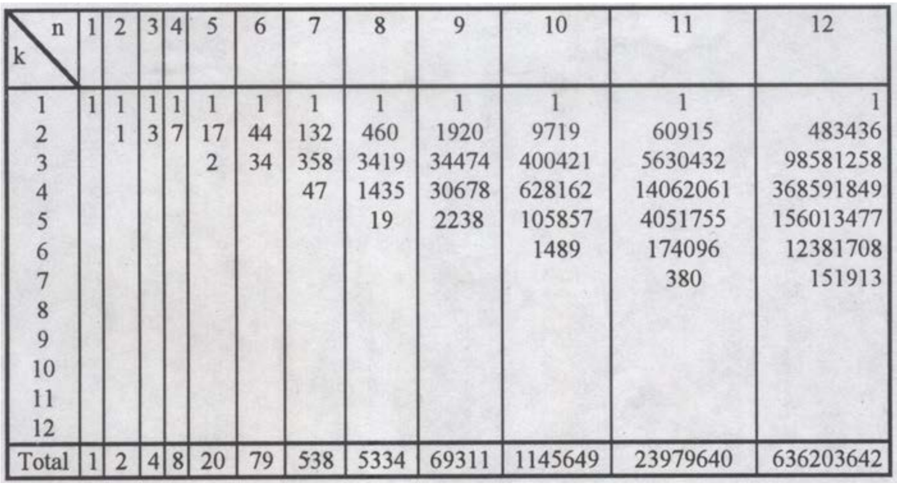
The numbers of non-isomorphic connected UPO n-element posets of height k.



k \ n	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0
2		1	2	4	10	27	88	328	1460	7799	51196	422521
3					2	32	324	3061	31055	365947	5230011	92950826
4							47	1388	29243	597484	13433899	354529788
5								19	2219	103619	3945898	151961722
6										1489	172607	12207612
7											380	151533
8												
9												
10												
11												
12												
Total	1	1	2	4	12	59	459	4796	63977	1076338	22833991	612224002

Table (7)

The numbers of non-isomorphic total UPO n-element posets of height k.



n k	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	17	44	132	460	1920	9719	60915	483436
3					2	34	358	3419	34474	400421	5630432	98581258
4							47	1435	30678	628162	14062061	368591849
5								19	2238	105857	4051755	156013477
6										1489	174096	12381708
7											380	151913
8												
9												
10												
11												
12												
Total	1	2	4	8	20	79	538	5334	69311	1145649	23979640	636203642

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