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ON THE RELATIVE FREQUENCY OF SERIES-PARALLEL TO N-FREE POSETS

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ABSTRACT.

Let s_n, f_n be the number of series-parallel partially ordered sets (posets) and N-free posets on n elements respectively. Mohring, (see[5], problem sessions pp.522-591), asked about the limiting ratio of s_n to f_n.

In this paper the authors prove constructively that the ratio s_n / f_n tends to zero as n tends to infinity.

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§1. INTRODUCTION.

Series-parallel posets and N-free posets have been found computationally attractive in different applications. These applications include, for example, electrical networks, several scheduling problems and sequencing problems.

The structural properties of special classes of posets have led to fas solution for nearly all problems that are known to be intractable on large classes of posets or graphs. Such classes include, the series-parallel case, N-free ordered sets, interval ordered sets and 2-dimensional posets.

In fact, this explains why considerable attention was focused on identifying special classes of precedence constraints for which the problem has a polynomial time solution.

§2. FUNDAMENTAL DEFN TIONS AND BASIC CONCEPTS.

In this section the basic definitions and fundamental composition operations, [2], have been introduced.

Let P_1 and P_2 be partially ordered sets (posets) with disjoint ground sets V_1 and V_2.

The *parallel composition,* $P_1 + P_2$, *(disjoint sum* or *disjoint union)* of' P_1 and P_2 is .the partial order on $V = V_1 \cup V_2$ whose ordering relation is defined as

$$u \leq v \ in \ P_1 + P_2 \quad iff \quad \left\{ \begin{array}{c} u, v \in V_1 and \ u \leq v \ in \ P_1, \\ or \\ u, v \in V_2 and \ u \leq v \ in \ P_2. \end{array} \right\} \quad (2.1).$$

The *series composition,* $P_1 \oplus P_2$, *(ordinal* sum) of P_1 and P_2 is the partial order on V whose ordering relation is defined as

$$u \leq v \ in \ P_1 \oplus P_2 \quad iff \quad \left\{ \begin{array}{c} condition \ (2.1) \ is \ satisfied, \\ or \\ u \in V_1 and \ v \in V_2 \, . \end{array} \right\} \quad (2.2).$$

The posets P_1 and P_2 are called respectively the *parallel blocks* and *series blocks* of' $P_1 + P_2$ and $P_1 \oplus P_2$.

A poset P is called *series-parallel* if it can be built. up from singleton orders by successive series and parallel compositions. Series-parallel posets constitute the smallest class of posets that contains the one-element poset and is closed under parallel and series composition.

A poset P is called N-free if its Hasse diagram does not contain an induced subgraph isomorphic to Fig.1.

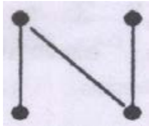


Fig. 1

Another characterization of' N-free posets, namely the CAC (chain-antichain-complete) property [3], is *"A poset is N-free iff each maximal chain. C, meets each maximal antichain, A"*.

In the next section we will use Stanley's relations that are developed in [6]. Hence we must recall the following terminologies.

Let X be a nonempty set of' isomorphically distinct finite posets such that no element P of' X is a disjoint union or an ordinal sum of' two nonempty posets. The elements of' X are called (+,⊕) *- irreducible posets. Stanley's technique aims to obtain the number of isomorphically distinct posets with n elements that can be* built up *from the elements of' X by the operations of' disjoint union* and ordinal sum. Henceforth, a poset that can be obtained in this way is called an X-poset... Hence, if P and Q are X-posets, then so are P + Q and P ⊕ Q. Moreover, the (+,⊕)-irreducible X-posets are simply the elements of X.

Let f'_n denote the number of X-posets with n elements. To count f_n, we consider two types of posets as follows. If' P is an X-poset and it can be written as $P_1 + P_2$ where neither P_i is empty, then P is called *essentially-parallel*. Similarly if' $P = P_1 \oplus P_2$ where neither P_i is empty, P is said to be *essentially-series*. Every X-poset not an element of X is either essentially-parallel or essentially-series, but not both. By convention every element of' X is considered both essentially-parallel and essentially-series.

Let a_n, u_n and v_n be the numbers of' (+_,⊕)-irreducible posets, essentially-parallel X-posets and essentially-series X-posets with n elements each respectively. Stanley defined the generating functions

$$F(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad A(x) = \sum_{n=0}^{\infty} i_n x^n,$$

$$U(x) = \sum_{n=0}^{\infty} u_n x^n \quad \text{and} \quad V(x) \sum_{n=0}^{\infty} v_n x^n$$

from which and Stanley's relations, [6]., one can obtain the following forms.

$$F(x) = \exp \left[\sum_{k\geq 1} V(x^k)/k \right] \qquad (2.3).$$

$$F(x) = (1 - U(x))^{-1} \qquad (2.4).$$

We deduce the following functional equation for the generating function V(x) from (2.3) and (2.4).

$$V(x) = 2 \left[\cosh \sum_{k\geq 1} V(x^k)/k - 1\right] + A(x) \qquad (2.5).$$

The main result of' Stanley is the theorem.

"F(x) satisfies the functional equation:

$$F(x) = \exp \left[\sum_{k\geq 1} 1/k \left(F(X^k) + 1/F(X^k) + A(X^k) - 2\right)\right] \qquad (2.6)."$$

§3. ANSWER OF MoHRING'S QUESTION.

In ([5], problem sessions pp. 522-591), Mohring asked about the limiting ratio of the number of' series-parallel posets with n elements to the number of' N-free posets with n elements.

To answer this question, we determine the asymptotic estimate for the number of posets specified in both cases.

Let U(x) $=\sum_{n=0}^{\infty} u_n x^n, V(x)=\sum_{n=0}^{\infty} v_n x^n and\ S(x)=\sum_{n=0}^{\infty} S_n x^n$, be the generating functions of essentially-parallel, essentially-series and total series-parallel posets respectively. Substituting A(x) = X in (2.6), we get (3.4).

$$S(x) = \exp\left[\sum_{k\geq 1}\frac{1}{k}\left(S(x^k)+\frac{1}{S(x^k)}-2+x^k\right)\right]. \quad (3.1)$$

This functional equation allows the determination of an asymptotic estimate for S_n. Using the technique of Bander ([2]), §7) one obtains, [6]

$$S_n \sim C\, n^{-3/2} \propto^{-n} \qquad (3.2).$$

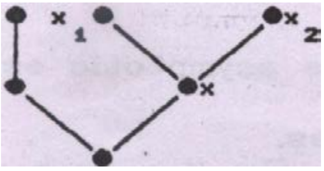
Where C is a constant and ∝ is the radius of convergence of S(x) which can be determined from the equation

$$S(\propto) = (1+\sqrt{5})/2 \qquad (3.3),$$

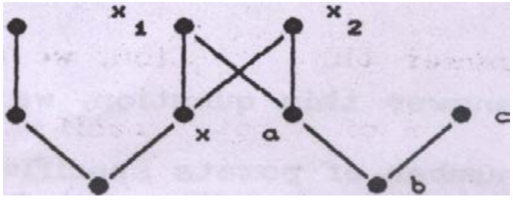
Or equivalently

$$1 = 2\,\mathrm{Sinh}\ \sum_{k=1}^{\infty} V(\propto^k)/k \qquad (3.4).$$

Now we construct a class *NF* of N-free posets whose generating function will turn out to have a radius of convergence strictly smaller than ∝. Towards this end we first, describe an operation on N-free posets. Suppose P is a connected N-free poset with at least. two elements. Choose a subset M ⊆ Max(P) which forms the set of upper covers of some element x ∈ P. Then construct a poset Q = Q[P] by adding to the elements of P three elements a , b and c such that b is covered by both a and c and a is covered by the elements of M, Fig.2 is an illustrative example.



P ; M=(x_1, x_2)



Q

Fi.g. 2

Clearly the poset Q[P] is N-free. However it is not series-parallel.

The class NF can be recursively defined as follows

(a) The single-element poset is in *NF.*

(b) If P_1 and P_2 are in *NF* then so are the posets $P_1 + P_2$ and $P_1 \oplus P_2$.

(c) If P is a connected poset in *NF* then the poset Q[P] is also in *NF.*

Let X be the class of (+,⊕)-irreducible *NF* poset Clearly the class *NF* is the smallest class containing X and is closed under taking disjoint unions and ordinal sums. In other words *NF* is the class of all X-posets So we can apply the results of Stanley [6] to *NF.*

Let $S^*(x) = \sum_{n=0}^{\infty} s_n^* \ x^n$ denote the generating function for NF-posets.

Furthermore let $V^*(\mathrm{x}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{v}_\mathrm{n}^* \mathrm{x}^\mathrm{n}, \mathrm{U}^*(\mathrm{x}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{u}_\mathrm{n}^* \mathrm{x}^\mathrm{n}$

and $\mathrm{A}^*(\mathrm{x}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{a}_\mathrm{n}^* \mathrm{x}^\mathrm{n}$ be the generating functions for NF posets which are , respectively, essentially-series, essentially-parallel and (+,⊕)-irreducible.

Note that an (+,⊕)-irreducible *NF* poset is either the single- element or the poset Q[P] for some essentially-series *NF* poset P.

Therefore :

$$A^*(x) = x^3 \quad (V^*(x) - x) + x \qquad (3.5)$$

Substituting in (2.5) we see that $V^*(x)$ satisfies the functional equation

$$V^*(x) = 2 \ [\cosh \sum_{k=1}^{\infty} V^*(x^k)/k - 1] + x^3(V^*(x) - x) + x \qquad (3.6)$$

Hence we have ,([2], theorem 5), the asymptotic estimate

$$V^* \sim C \ n^{-3/2} \beta^{-n} \qquad (3.7),$$

where C is a constant and β is the radius of' convergence of $V^*(x)$ which is determined by the equation

$$1 - \beta^3 = 2 \ \sinh \sum_{k=1}^{\infty} V^*(\beta^k)/k .$$

Now every series-parallel poset is an *NF* poset. Therefore $v_n \le v_n^*$, that is, $V(x) \le V^*(x)$ is satisfied for all $0 \le x < \beta$. Hence

$$2 \sinh \sum_{k=1}^{\infty} V(\beta^k)/k \le 2 \sinh \sum_{k=1}^{\infty} V^*(\beta^k)/k = 1-\beta^3$$

Thus

$$2 \sinh \sum_{k=1}^{\infty} V(\beta^k)/k < 2 \sinh \sum_{k=1}^{\infty} V(\alpha^k)/k$$

which implies that $\beta < \alpha$. Now the function $S^*(x), U^*(x)$ have the same radius of convergence β which implies that

$$S_n^* \sim C' \quad n^{-3/2} \beta^{-n}.$$

Then

$$\lim_{n \to \infty} \frac{s_n}{s_n^*} = 0,$$

which shows that almost all N-free posets are not series-parallel.

APPENDIX.

In this appendix we present the number of unlabeled N-free, f_n with n elements and series-parallel posets, s_n, with n elements, n ≤ 12. Also, the values s_n / f_n are present.ad and show that the ratio, s_n / f_n is rapidly decreasing even for the small values of n.

Finally, the authors point out that the number of N-free posets disagrees, for n = 10, with that given by Mohring, [3], who gave the numbers only for n $\leq$ 10.

f_n : N-free posets. f_n^c : connected N-free posets.

s_n: series-parallel posets. s_n^c: connected series-parallel posets.

Table 1

n	f_n^c	f_n	s_n^c	s_n	s_n/f_n
1	1	1	1	1	1
2	1	2	1	2	1
3	3	5	3	5	1
4	9	15	9	15	1
5	31	49	30	48	0.979591836
6	115	180	103	167	0.927777778
7	.474	715	375	602	0.841958041
8	2097	3081	1400	2256	0.732229795
9	9967	14 17	5380	8660	0.609129914
10	50315	69!105	21073	33958	0.485773549
11	268442	363926	83950	135292	0.371756895
12	1505463	1996922	338878	546422	0.273632119

Notice that :

The number of N-free posets was calculated through a computer program, [1], which generates prime N-free posets taking into account. the isomorphism. The number of posets increased rapidly when n increased, so did the running time. For example, at n = 11, the recorded running time is about. 105 minutes and, at n = 12, it is near 18 hours. The authors expect that the running time for n = 13 will be more than 200 hours.

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بســـــم اللــه الـــرحـــمـــن الـــــرحـــيــــم

عن نسبة عدد الفئات المرتبة جزئيا
ذات خاصية التوالى-التوازى الى مثيلتها الخالية من الشكل N

بـيـومــى ابــراهــيـم بــيــومــى – مـحـمـد حـامـد الـزهـار – سـهـيـر مـحـمـد خـمـيـس
قــسـم الـــريــاضــيــات – كــلــيـة الــعـلـــوم – جــامــعــة عــيـــن شــمـــس

مـلـخـص البـحـث

فى هذا البحث يتعرف الباحثون الى سؤال مورنج فى ([5] , problem sessions pp. 525-591) عن حدود النسبة s_n/ f_n حيث ان "s_n" هى عدد الفئات المرتبة جزئيا ذات n من العناصر ولها خاصية التوالى - التوازى (series-parallel) ومثيلتها "f_n" الخالية من الشكل "N" (N-free) وقد أثبت فى هذا البحث بطريقة بنائية ان هذه النسبة s_n/f_n ⟵ الصفر عندما n ⟵ المالانهاية .

كلية العلوم

النشـــــرة العلميـــه
لجامعـــــة عين شمس

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