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ENUMERATION OF 2-DIMENSIONAL Posets VIA COUNTING PRIME 2-DIMENSIONAL POSET

By

Bayoumi B.I. El Zahar M. H. Soheir M.K.

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Enumeration Of 2-Dimensional Posets Via Counting Prime 2-Dimensional Posets

BAYOUMI I. BAYOUMI. MOHAMED H. EL-ZAHAR AND SOHEIR M. KHAMIS

Department of Mathematics

Faculty of Science

Ain Shams University, Cairo, Egypt.

Abstract.

Let f_n be the number of pairwise non-isomorphic 2-dimensional posets on n elements. Applying results or Stanely and Polya, we obtain a functional equation for the generating function $\Sigma f_n \ X^n$ in terms or the generating function for prime 2-dimensional posets.

We then describe the techniques used in calculating the number of prime posets of dimension two. Also, it is proved that the previously known number for n=10, given in [5] 1985, is not correct.

AMS Mathematics subject classification (1980).

05A15 and 06A10.

1- Introduction.

In this paper, we describe a computer program for counting the number of pairwise non- isomorphic 2-dintensional partially ordered sets (posets) on n elements.. It is known [3] that this number is asymptotically equal to $\frac{1}{2}$ n! and so increases very rapidly. Therefore, special techniques had to be devised for calculating these numbers even for small values of n. The main idea of these techniques is as follows . Well known results of Stanely [6] and Polya [4] allow the counting of 2-dimensional posts from the numbers of prime 2-dintensional posts . So the computer program concentrates on finding the number of prime2-dimensional posets.

It took 4 hours of personal computer time to calculate the number of these posets for n $\leq$ 10 .
We include the results of these computations. As it turns out, the previously known numbers for various classes of posets for n = 10, given in [5], seem to be not correct. Some corrections to these numbers were given in [1].

2-Definitions and Basic Concepts.

A *partially* ordered set , poset, is a pair $(P,\leq)$ with a nonempty set P and a partial ordering, $\leq$ on *P.* For simplicity we ·denote the ordered set, $(P,\leq)$ By its ground set P. A *linear extension* L of P is a total ordering of the elements of P such that *$x \leq y$ in P implies that $x \leq y$ in L.* The *dimension d(P) of a poset P is the minimum number of linear extensions whose intersection is the partially ordering of P, [2]. Posets P with d(P) ≤ 2 are of special interest.*

A representation of 2-dimensional poset P is a pair $\{L_1, L_2\}$ of two linear extensions L_1, L_2 whose intersection is the ordering of P, and it will be denoted by (L_1, L_2) . if the Order of L_1 *and* L_2 matters. The poset P is called *uniquely representable,* [3], If *it* has a *unique representation* $\{L_1, L_2\}$. P is called *uniquely representable up to isomorphism if whenever* (L_1, L_2) *and* (L_3, L_4) *are* two *representations* of P then (L_3, L_4) is induced from (L_1, L_2) by an automorphism of P.

Representations (L_1, L_2) of a poset P of n elements are closely related to permutations on n elements. This relation was discussed in details in [3]. If we denote the elements of P in the same Order as they appear in L_1 by 1,2,...n then L_2 is merely a permutation of 1,2,... , n. This is illustrated in the example of figure 1.

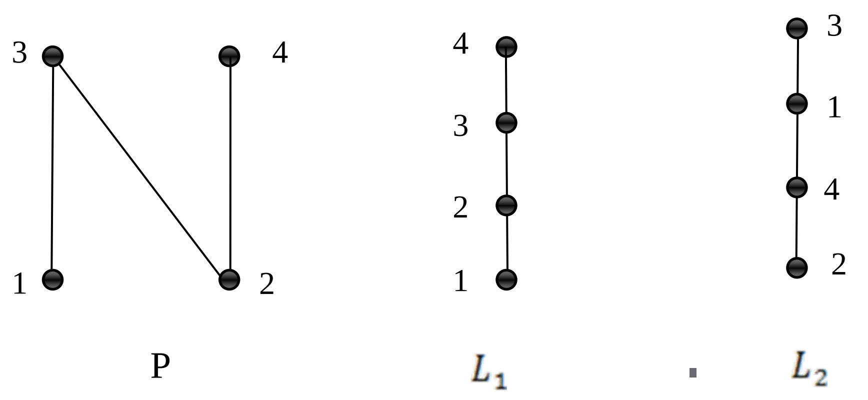

Fig. 1 a poset P and its representation (L_1, L_2).

Let us denote the permutation obtained in this way by $\sigma(L_1,L_2)$ which equals L_1,L_2^{-1} whenever each linear extension L of P can be viewed as a bijection $L : P \to \{1,2,\ldots,n\}$, [3]. Thus any counting of 2-dimensional posets has to rely on classifying permutation of 1,2,...,n. However the number of such permutations, being n!, increases rapidly with n. Even more, a poset can have more than one representation and could correspond, therefore, to several permutations. For example interchanging the roles of L_1 and L_2 in the above example, we get another permutation $\sigma(L_2,L_1) = 3142$ which is the inverse of $\sigma(L_1,L_2) = 2413$.

Due to these complications we will concentrate on counting the so called prime 2-dimensional posets.

3-Prime 2-Dimensional Posets.

Let A be a subset of elements of a poset P. A is called *P-autonomous* if for every $a_1,a_2 \in A$

And $b \in P-A$, we have

(i) b $< a_1$ iff b $< a_2$,

(ii) $a_1 <$ b iff $a_2 < b$.

P is called *decomposable* if it has a non-trivial autonomous set A (*i.e.,* $1 < |A| < |P|$). Otherwise, P is called *indecomposable or prime*.

Autonomous sets can also arise by the Substitution composition . Let $x \in P$. We can replace x by a poset Q such that for every $y \in Q$ and $z \in P-x$: $z < y$ iff $z < x$ and $y < z$ iff $x < z$. This gives rise to a poset P in which Q is P-autonomous. Conversely we can reduce an autonomous set to single element. If this is done to every maximal autonomous set in P we get a unique prime post called the *prime image* of P.

Now, assume that P is of dimension 2 and let (L_1, L_2) be a representation of P. A subset. $X \subseteq P$ is said to *appear consecutively* in (L_1, L_2) if no element of P-X occurs between any two elements from X in either L_1 or L_2. Obviously every subset $X \subseteq P$ which appear consecutively on a representation of P must be a P-autonomous. Conversely, every maximal P-autonomous set must appear consecutively in every representation of P.

It is true that different representations of the same poset P are obtained by interchanging their restrictions to some P-autonomous sets. In order to overcome this difficulty, it is sufficient to consider prime 2-dimensional posets which are uniquely representable. (theorem 1 in [3]).

However, to explain how to get 2-dimensional posets from prime ones. we have first to consider the operation of disjoint union and ordinal sum.

Let P and Q be posets. Regard P and Q as relation on tow disjoint Set X and Yrespectively. The disjoint*union* P + Q is defined to be the partial ordering on $X \cup y$ satisfying:

(1) if $x \in X, y \in X$ and $x \leq y$ in P, then $x \leq y$ in P+ Q ,

(2) if $x \in Y$, $y \in Y$ and $x \leq y$ in Q ,then $x \leq y$ in P+Q.

The *ordinalsum* $P \oplus Q$ is defined to be thepartial Ordering on $X \cup Y$ satisfying (1),(2) and the additiona1 condition .

(3) if $x \in X$ and $y \in Y$, then $x \leq y$ in $P \oplus Q$. P is called $(+,\oplus)$-*irreducible* poset ifit cannot be constructed by a disjoint union or an ordinal sum of two nonempty posets.Figure 2, illustrates the P + Q , $P \oplus Q$ and $(+,\oplus)$ -irreducible posets.

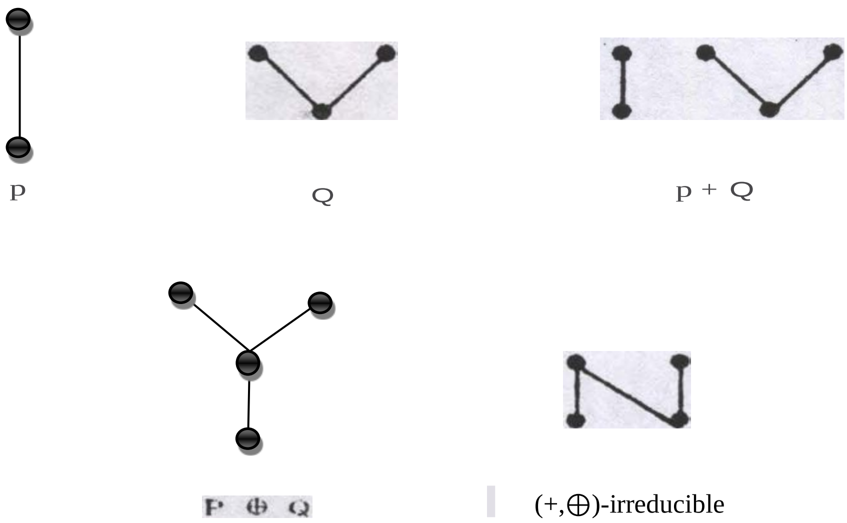

Fig .2

Now, we have the following useful properties of the 2-dimensional posets.

Theorem 1.

(1) If P_1 and P_2 are 2-dimensional posets then so are the posets P_1+P_2 and $P_1\oplus P_2$.

(2) If P and Q are 2-dimensional posets then the poset P, obtained from rep1acing x ∈ P by Q, is also 2-dimensional poset.

Proof.

(1) Let P_1 and P_2 be 2-dimensional posets with representation (L_1, L_2) and (L_3, L_4) respectively.

From the structure of P_1+P_2 and $P_1\oplus P_2$, the elements of P_1 or P_2 must appear consecutively as follows. L_1 must appear above (below) L_3 or L_4 and in the same way L_2 must appear below (above) L_4 or L_3 respectively, in the resulting linear extensions of $P_1\oplus P_2$, we must connect L_1 and L_2 above (below) L_3 and L_4 respectively.

Therefore, the minimum number of linear extensions whose intersection is the partially ordering of P_1+P_2 or $P_1\oplus P_2$ is equal to 2.

(2) Since P is a 2-dimensional poset with a representation (L_1, L_2), then if we insert a representation of Q instead of x in both L_1 and L_2 we obtain a representation of P. Thus P has dimension 2.
This completes the proof.

A poset P is said to be *connected* if its comparability graph is connected, i.e. it has one component, otherwise P is said to be *disconnected.*

Any 2-dimensional poset, P can be decomposed as follows:

(1) If P is not $(+,\oplus)$-irreducible poset then it must be the disjoint union $P_1 + P_2$ or the ordinal sum $P_1 \oplus P_2$

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of some posets P_1, P_2. Repeating this process, gives $(+,\oplus)$-irreducible posets.

(2) If P is an $(+,\oplus)$-irreducible poset, then P-autonomous sets can be reduced to a single element. Applying this reduction to every maximal autonomous set in P we get the prime image of P.

Therefore, any 2-dimensional poset P can be built up from prime 2-dimensional posets, by using substitution composition and then the operations of disjoint union and ordinal sum.In the following section we discuss this process· in terms of generation functions.

4. Generating Functions.

Let f_n denote the number of 2-dimensional Posets of n elements. Define the generating function F(X) = $\sum_{n=o} f_n X^n$, with f_o=1. Let i_n denote the number of $(+,\oplus)$-irreducible posetswith n elements. Let u_n, v_n denote the number of parallel and series 2-dimensional posets with n ·

elements respectively. We can consider (+,⊕)-irreducible 2-dimensional posets as being both series and parallel. Hence $f_n = v_n + u_n - i_n$, for n ≥1.

$I(x)=\sum_{n=0}^{\infty} i_n x^n$, $U(x)=\sum_{n=0}^{\infty} u_n x^n$, $V(x)=\sum_{n=0}^{\infty} v_n x^n$.

According to Stanely , [6], these generating functions satisfy:

$$F(x) = U(x) + V(x) - I(x) \quad (1).$$

$$F(x) = \exp\left(\sum_k V(x^k)/k \right) \quad (2)$$

$$F(x) = \frac{1}{1 - U(x)} \quad (3).$$

Put

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{k=1}^{\infty} V(x^k)/k \quad (4).$$

Equating the coefficients in both sides we easily obtain

$$na_n = \sum_{d/n} d\, v_d \quad (5),$$

and equation (2) becomes

$$\sum_{n=0}^{\infty} f_n x^n = \exp\left[\sum_{p=1} a_p x^p \right] \quad (6).$$

The numbers f_n and a_n are then related by a Well known identity

$$nf_n = na_n + \sum_{p=1} ka_k f_n - k \quad (7)$$

which can be obtained by differentiating both sides of (6) w . r .t x and equating coefficients.
Similarly from (3) and (4) we get

$$1-\sum_{n=0}^{\infty} u_n x^n = \exp[-\sum_{p-1} a_p x^p] \qquad (8).$$

and the corresponding relation between a_n and u_n is

$$nu_n = na_n - \sum_{k=1}^{n-1} ka_k u_n - k \qquad (9).$$

Again from (3) We can easily deduce that

$$f_n - u_n - u_{n-1}f_1 - u_{n-2}f_2 - \cdots - u_1 f_{n-1} = 0,$$

that is

$$U_n = \sum_{i=1}^{n-1} u_i f_{n-i} + i_n \qquad (10).$$

It is clear that we can determine the values of f_n, v_n and u_n Provided we know the values or i_n.

In the following special techniques for computing the umbers i_n and the numbers, P_n, of prime 2-dimensional posets.

Theorem 2.

If P is a prime 2-dimensional poset then t he automorphism group of P, Γ(P) ,satisfies $|\Gamma(P)| \leq 2$.

Proof.

Assume that $\sigma \in$ Γ(P). Let(L_1, L_2) denote the unique representation of P. Applying σ to (L_1, L_2) we get another repesentation $(L_1\sigma, L_2\sigma)$. Since P is uniquely representable we have either

$(L_1\sigma, L_2\sigma) = (L_1, L_2)$ or $(L_1\sigma, L_2\sigma) = (L_2, L_1)$.

In the first case σ is the identity and in the latter case σ has order 2. In any case $|\Gamma(P)|$ has at most two elements.

Definition.

Let P be a 2-dimensional poset with $|\Gamma(P)| = 2$. An element $x \in P$ is said to be a *symmetric element* if its orbit under $\Gamma(P)$ has two elements. Otherwise x is called a *fixed* element. For example, figure 3 illustrates a poset P, which has one fixed element and four symmetric elements.

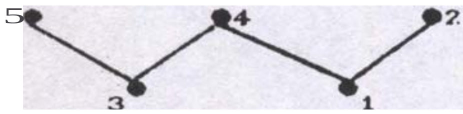

Fig .3 4 is a fixed element

1.2.3, 5 are symmetric elements.

The symmetric elements give rise to isomorphic posets if we use substitution composition. Let x, y $\in$ P are two symmetric elements, if either x or y is replaced by a poset Q then the tow resulting posets are the same. In the above example if we replace the element 2 or 5 by antichain of 2-element then two resulting posets are isomorphic to the poset shown in figure 4.

Fig.4

To overcome this complication,Polya's enumeration theorem [4,Ch.2] may be applyed .

Let P. be a prime 2-dimensioua1 poset with k fixed elme11ts and 2m symmetric elements and let $\Gamma(P)$ be its automorphism group, the cycle index of $\Gamma(P)$ is

$$Z(\Gamma(P)) = s_1^n + s_1^n s_2^m, n = k + 2m \qquad (11)$$

Let $g_p(x) = \Sigma\, g_{pq} x^q$ be the generating function for all posets whose prime image is P. According to Polya's theorem, we have

$$g_p(x) = \frac{1}{|\Gamma(P)|} Z(\Gamma(P), S_i \to F(x^i)) \qquad (12).$$

Using theorem (2) and equations (11), (12) we obtain

$$g_p(x) = \frac{1}{2}(F^n(x) + F^k(x)F^m(x^2)) \qquad (13).$$

Therefore,

$$I(x) = 1 + \frac{1}{2} \sum_{n=2}^{\infty} \sum_p (F^n(x) + F^k(x)F^m(x^2)) \qquad (14).$$

Note that, the second summtion is taken over all prime 2-dimensional posets with n elements which will be determined in the following section.

5. Special Techniques for Computing p_n-

In this section we introduce four algorithms for counting the number or prime 2-dimensional posets with n elements, n ≥ 4 . These algorithms

depend on the basic concepts and fundamental theorems given above.

The program creates a permutation of n and decides whether it represents a prime 2-dimensional poset, (prime permutation) or not .

Now , algorithm (1) creates permutations V of n in a lexicographic order . Initially, V = 1,2,3, . . . n. To find the successor permutation of $v_1, v_2, v_3, \ldots, v_n$ we search for max i such that $v_{L+1} > V_L$ if there exist no such 1 then algorithm ends . We then replace V_L bythe minimum element of A where A = (v_j: j > i and $v_j > v_i$) . Then we rearrangethe remaining elementsin ascendingorder . We use a logical identifier , Done .which becomes true when we arrive at the last permutation .

Algorithm (I)

"Create permutations in lexicographic order"

begin

for $i \leftarrow 1$ *to* n *do*

$V[i] \leftarrow i$

Done $\leftarrow$ *false*

```
While not (Done) do
    A ← [1..n]
    i ← n+ t
    Repeat
        i ← i - 1
    Until  ( V[i + 1] > V[ i] ) Or ( i= 0 )
    If   i ≠ 0
        then
          for j ← 1 to i do
                    A←A –{ V[j ]}
          m←min {i : i ∈A ∩{V [i] +1..n}}
          A ←A ∪ {V[i] – [m]
          V[i] ←m
        for j←  i+1 to n  do
           m ←min{i : i ∈ A}
        V[j ] ← m
           A ← A . . . (m)
      else
                    Done ← true
  end.
```

This algorithm creates some unneeded permutations which increases the running time. So, we used the following modifications on algorithm (1) to omit these permutations. Firstly, we exclude all permutations that begin with the digit one because in this case the corresponding poset is not prime. After that, start with the permutation (1 , n, n-1 ,...,2) which reduces the number of permutations by n-1,The algorithm also excludes all permutations that begin with the digit n because these permutations represent 2-dimensional posets that contain at least two components one of which is the isolated element n .So, the last permutation must be equal (n- 1,n,n-2,...,1).This also reduces the number of permutations by n-1.

Secondly, algorithm (1) excludes all permutations having two successive digits that appear consecutively, i.e., $|V_{L+1} - V_L| = 1, 1 \leq i \leq n-1$.

Because these permutations represents 2-dimensional posets that have an autonomous set of two elements.

This can be done as follows:

If $|V_{i+1} - V_i| = 1$ **then**
$A \leftarrow \{1,\ldots,n\} - \{V_1, \ldots, V_L\}$
for j←n **downto** i+1 **do**
$V_j \leftarrow \min\{k : k \in A\}$
$A \leftarrow A - \{V_j\}$

This modification reduces the number of permutations significantly.

Now, algorithm (2) checks if a 2-dimensional poset is prime or not .This test is easy if we take into account the permutation representing this poset. If the elements of the k- tuple ($V_{j+1}, V_{j+2}, \ldots, V_{j+k}$) appear consecutively, this is $|V_r - V_s| \geq 1 \; \forall$ r,s ∈ (j+1,j+2, ...,j + k) ,then k-tuple represents an autonomous set ,therefore the permutations is not prime.

Algorithm(2)

"Test a Permutation represents a prime poset"

begin

Decision1 ← false

for i ← l to n-1 do

for k ← to n - i do

if k· = n-1 then get next i

for j ← i to $i+k-1$ do

for i ← j+ 1 to $i+k$ do

if $|V[i]-V[j]| > k$ then get next j

stop (the permutation is not prime)

Decsion1 ← true

end.

The identifier Decision1 returns by the value true if a poset is prime . Cleary, a prime poset is connected so 1 that we can reduce the number of permutations more and more by excluding all permutations representing disconnected posets. This can be

done using algorithm (3). In this algorithm, for any k if there is no j ∈ { $V_{n-k},\ldots,V_n$} such that $k < i$ ∀ k ∈ {1,2,…,n} then the elements k and j are comparable ∀ j ∈ { $V_{n-j},\ldots,V_n$ } , and therefore, the permutation representing the poset, P. has at least two components. Otherwise P contains one component, i.e. connected. Algorithm (3) plays an important role in reducing the running time of the main program.

Algorithm (3)

"Test a permutation represents a connected poset"

begin

```
    Decision2 ← false
    for k ← 1 to n-1 do
        j ← n+ 1
        i ← n-- k
        Repeat
            J ← J-1
    until (V[j] J> K) or (j=i)
    if J = i then
        stop( the permutation
            is disconnected)
    Decision2 ← true
    end.
```

The above three algorithms together are used to determine the number of' prime 2-dimensiona L poset.s or n elements. n ≥4. To get the number of nonisomorphic prime 2-dirnensiona l poset. i t is sufficient to show that. permutations corresponding to prime 2-d.irnensional posets are self inverse. If a permutataion is a self inverse then the corresponding poset. will appear once in the list of permutation. So, it is counted by one. If a permutation , V, is not a self inverse ($v \neq V^{-1}$)then there exist two posets P and p, corresponding to V and V^{-1} respectively which are isomorphic. So , P will appear and counted by half .In algorithm (4), the condition of self inverse which is

v_i= J ↔ V_1=i ∀.1∈{1,2, … , n}

Will be tested.

Algorithm (1)

"Test . a permutation is inverse "

begin

Decision 3 ⟵ false

for i ⟵ 1 to n do

j ← V[i]

if V[i] ≠ i **then**

stop (the permutation is not self inverse)

Decision3 ← true

K ← 0

For i ←1 **to** n **do**

If v[i] = i **then**

K ← k + 1

end.

The final program consists of the following steps and its results are

l_k : the number of prime 2-dimensiona l posets with

K fixed elements that denoted by P(n,k). see table 1.

count : the number of prime 2-dimensional posets of n elements, that denoted by p_n,

see table 2.

Step 1 .

get first permutation , put count =0.

• .

step 2.

get next permutation if it exists, else go to

step 3.

test the poset for connectedness , (algorithm 3 .) .

if it is connected then go to step 4, else go to step 2 .

Step 4

test whether the poset is prime, (algorithm 2.

if it is ..... prime then go to step 5. go to

step 2.

Step5

the permutation for begin self inverse. If it.

self inverse then put count ← count + 1

$L_k \leftarrow l_k + 1$. Otherwise count ← count + 0.5,

$L_k \leftarrow l_k + 0.5$. go to step 2.

step 7.

stop .

The results of these computations for n ≤ 10 are giveni n the following tables.

p(n,k) : number of prime 2-dimensional posets with

n elements and k fixed elements.

Table 1

n \ k	4	5	6	7	8	9	10
0	0	0	1	0	8	0	83
1	0	2	0	9	0	90	0
2	0	0	3	0	28	0	351
3	0	0	0	1	0	36	0
4	1	0	0	0	0	0	30
5		2	0	0	0	0	0
6			21	0	0	0	0
7				164	0	0	0
8					1445	0	0
9						14010	0
10							149036

P(n , k)

P_n : number of prime 2-dimensional posets

with n elements

i_n: number of $(+,\oplus)$-irreducible 2-dimensional posets

with n elements.

p_n: number of connected 2-dimensional posets with n elements.

f_n : number of 2-dimensional posets with n elements .

Table -2.-

n	p_n	i_n	v_n	f_n
1	0	1	1	1
2	0	0	1	2
3	0	0	3	5
4	1	1	10	16
5	4	12	44	63
6	25	101	235	315
7	174	876	1564	1956
8	1481	8105	12399	14794
9	14136	81678	113936	131526
10	149490	895498	1179392	1331848

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