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Tit-For-Tat and Win Stay-Lose Shift strategies via memory-two

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Abstract. Decisions in many dilemmas are based on a combination of factors, including as incentive, punishment, reputation, and memory. The impact of memory information on cooperative evolution in multi-round games is a decision-making process in group evolution. The iterated prisoner's dilemma is an excellent model for the development of cooperation amongst the payoff-maximizing individuals. Since tit-for-tat proved successful in Axelrod's repeated prisoner's dilemma tournaments, there has been a great deal of interest in creating new strategies. Every iterative prisoner's dilemma method bases its decision-making on a specific duration of past contacts with the opponent, which is referred to as the memory's size. This study examines the impact of strategy memory size on the evolutionary stability of n-person iterated prisoner's dilemma strategies. In this paper, we address the role that memory plays in decision-making. We interested in the model of the Iterated Prisoner's Dilemma game for three players with memory two, and we will look at strategies with similar behavior, such as Tit-For-Tat (TFT) strategies as well as Win Stay-Lose Shift (WSLS) strategies. As a result of this paper, we have shown that the effect of memory length is almost non-existent in the competitions of strategies that we studied.

Keywords: Memory-Two, Tit-For-Tat strategies (TFT), three-players iterated prisoner's dilemma game (3P-IPD), transition matrix, Win Stay-Lose Shift strategies (WSLS)

1. Introduction

Game theory does not tell us how to play the game; rather, it discusses some possible strategies for playing that you could find appealing. Additionally, the area of applied arithmetic is utilized to develop the best possible approach for contending with uncertainty and inadequate knowledge like most real-life scenarios. It is the mathematical analysis of conflict modeling and decision-making that occurs in all kinds of fields and companies on a daily basis.

Since the game theory is the theory of "trategic thinking", it has been developed for military purposes and defense. In the past, it has also been used as an alternative and complementary approach to deal with robustness in the presence of worst-case uncertainties or disturbances in many areas such as economics [1, 4], engineering [5], computer science, politics [8, 11, 14, 15], and sociology. As well as to the biological sciences such as evolutionary biology [3, 7, 12]. Moreover, scientists tend to use game-theoretic tools to avoid traffic flows or predict blackouts in power networks. In the 21^{st} century, game theory applies to a wide range of behavioral relations.

The mathematician John Von Neumann and the economist Oskar Morgenstern wrote a book [6] in 1944 that contained the first mention of game theory. This book's second edition offered an axiomatic theory of expected utility that enabled economists and statisticians to study decision-making in the face of uncertainty. In the 1950s, a large number of academics greatly expanded the field of game theory. In the 1970s, it was specifically used in relation to evolution.

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There are two types of games: simultaneous games [2, 9] and alternating games [10, 13]. In a simultaneous game, both players make their choices in the same time without knowing the other player's choice. While in the alternating game, the first player makes his choice and then the other player makes his choice after knowing the first player's choice. In this paper, we will discuss one of these two types, which is the simultaneous game through a three-player competition model in the iterative game.

The Prisoner's Dilemma is one of the most popular and simple strategy models developed by Merrill Flood and Melvin Dresher in 1950. In this model, no matter how many participants, each participant has only two decisions. Either he chooses to cooperate and plays C or defect and then plays D. Accordingly, in two-player Iterated Prisoner's Dilemma games, we have four outcomes (C, C), (C, D), (D, C) and (D, D), so there are 2^4 possibilities for the transition arrows made a total of 16 different strategies denoted by $S_0, S_1, ..., S_{15}$. Whereas, in three-player Iterated Prisoner's Dilemma games, we have eight outcomes $(C, (C, C))$, $(C, (C, D))$, $(C, (D, C))$, $(C, (D, D))$, $(D, (C, C))$, $(D, (C, D))$, $(D, (D, C))$ and $(D, (D, D))$. In our study, we look at symmetric games, so the outcomes become six only $(C, (C, C))$, $(C, (C, D))$, $(C, (D, D))$, $(D, (C, C))$, $(D, (C, D))$ and $(D, (D, D))$. Wherefore, there are 2^6 possibilities for the transition arrows made a total of 64 different strategies denoted by $S_0, S_1, ..., S_{63}$. We can categorize strategies based on their behavior, such as Tit-For-Tat, Win Stay-Lose Shift Strategies and other Strategies.

Tit for tat is a highly effective strategy in game theory. An agent using this strategy will first cooperate, then repeat opponent's previous decision. If the opponent previously was cooperative, the agent is cooperative. If the opponent previously was defective, the agent is defective. In Win Stay-Lose Shift strategies, if the game in the previous round has a succeeds, then the player plays the same strategy in the next round. Alternatively, if the game resulted in a failure, the player switches to another action.

Previous studies focused on studying the repeated prisoner's dilemma game for two players only, and recently we studied the repeated prisoner's dilemma game for three players but with memory-one [2,17], meaning that the player relies in his decision on the previous round only. In this paper, we will discuss the repeated prisoner's dilemma game for three players but with memory-two. We will focus on Tit-For-Tat and Win Stay-Lose Shift strategies and study the competition between them. At the end of this paper, we will answer the question of what does memory two do in the competition of Tit-For-Tat strategies with each other, and also the competition of Win Stay-Lose Shift strategies with each other.

2. Payoff and transition matrices

We will study the simultaneous game on Iterated Prisoner's Dilemma model. We assume that both players play simultaneously without knowing the other player's decision. Both players get a reward, $\mathcal{R}$, when they cooperate and get a punishment, $\mathcal{P}$, when they defect. While if one of them cooperates and the other defect, then the defector gets $\mathcal{T}$ and the cooperator gets $\mathcal{S}$. So, the corresponding payoff matrix for (2P-IPD) [9] is given by

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} \mathcal{R} & \mathcal{S} \\ \mathcal{T} & \mathcal{P} \end{bmatrix} \end{array}, \tag{1}$$

where:

$$\mathcal{S} < \mathcal{P} < \mathcal{R} < \mathcal{T} \; and \; \mathcal{R} > \frac{\mathcal{T} + \mathcal{S}}{2}. \tag{2}$$

Based on the aforementioned, we have four outcomes $O_1 = (C, C)$, $O_2 = (C, D)$, $O_3 = (D, C)$ and $O_4 = (D, D)$ and if we assume that player I plays with probability $P = (p_1, p_2, p_3, p_4)$ and player II with probability $Q = (q_1, q_2, q_3, q_4)$, where p_i is the probability of player I for playing C after outcome O_i. Therefore, the Markov transition matrix for two players [9] is given by

$$M_2 = \begin{bmatrix} p_1 q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2 q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3 q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4 q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{bmatrix}, \tag{3}$$

where $p_1(1 - q_1)$ is the transition probability from state O_1 to O_2, where player I plays C by probability p_1 after state O_1 and $(1 - q_1)$ probability of player II playing D after state O_1.

As well, in Three-player Iterated Pensioner's Dilemma games, the player's payoffs are $\mathcal{R}, \mathcal{K}, \mathcal{S}, \mathcal{T}, \mathcal{L}$, or $\mathcal{P}$ and the correspondence payoff matrix for (3P-IPD) can be represented by

$$\begin{array}{c|ccc} & CC & CD & DD \\ \hline C & \mathcal{R} & \mathcal{K} & \mathcal{S} \\ D & \mathcal{T} & \mathcal{L} & \mathcal{P} \end{array}, \tag{4}$$

where

$$\mathcal{S} < \mathcal{P} < \mathcal{K} < \mathcal{L} < \mathcal{R} < \mathcal{T}. \tag{5}$$

The three-player Iterated Prisoner's Dilemma, we have six outcomes $O_1 = (C, (C, C))$, $O_2 = (C, (C, D))$, $O_3 = (C, (D, D))$, $O_4 = (D, (C, C))$, $O_5 = (D, (C, D))$ and $O_6 = (D, (D, D))$ and if we assume that player I plays with probability $P = (p_1, p_2, p_3, p_4, p_5, p_6)$, player II with probability $Q = (q_1, q_2, q_3, q_4, q_5, q_6)$ and player III with $W = (w_1, w_2, w_3, w_4, w_5, w_6)$. Therefore, the Markov transition matrix for three-player [17] is given by

$$M_3 = \left(\begin{array}{llll} p_1q_1w_1 & p_1[q_1(1-w_1)+(1-q_1)w_1] & p_1(1-q_1)(1-w_1) & (1-p_1)q_1w_1 \\ p_2q_2w_4 & p_2[q_2(1-w_4)+(1-q_2)w_4] & p_2(1-q_2)(1-w_4) & (1-p_2)q_2w_4 \\ p_3q_5w_5 & p_3[q_5(1-w_5)+(1-q_5)w_5] & p_3(1-q_5)(1-w_5) & (1-p_3)q_5w_5 \\ p_4q_2w_2 & p_4[q2(1-w_2)+(1-q_2)w_2] & p_4(1-q_2)(1-w_2) & (1-p_4)q_2w_2 \\ p_5q_3w_5 & p_5[q_3(1-w_5)+(1-q_3)w_5] & p_5(1-q_3)(1-w_5) & (1-p_5)q_3w_5 \\ p_6q_6w_6 & p_6[q_6(1-w_6)+(1-q_6)w_6] & p_6(1-q_6)(1-w_6) & (1-p_6)q_6w_6 \end{array} \right.$$

$$\left. \begin{array}{ll} (1-p_1)[q_1(1-w_1)+(1-q_1)w_1] & (1-p_1)(1-q_1)(1-w_1) \\ (1-p_2)[q_2(1-w_4)+(1-q_2)w_4] & (1-p_2)(1-q_2)(1-w_4) \\ (1-p_3)[q_5(1-w_5)+(1-q_5)w_5] & (1-p_3)(1-q_5)(1-w_5) \\ (1-p_4)[q_2(1-w_2)+(1-q_2)w_2] & (1-p_4)(1-q_2)(1-w_2) \\ (1-p_5)[q_3(1-w_5)+(1-q_3)w_5] & (1-p_5)(1-q_3)(1-w_5) \\ (1-p_6)[q_6(1-w_6)+(1-q_6)w_6] & (1-p_6)(1-q_6)(1-w_6) \end{array} \right). \tag{6}$$

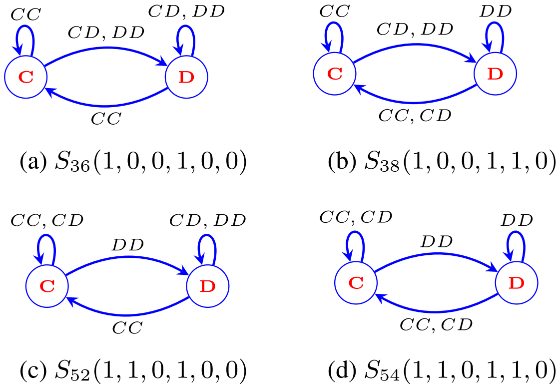
3. Strategies competitions with memory two

The purpose of this section is to explain the significance of the Tit-For-Tat and Win Stay-Lose Shift strategies with memory-two in the context of the three-player Iterated Prisoner's Dilemma game. A two-state automata is adopted to represent the strategies where each player is automaton which can be in one of two states, namely C and D. The sixty-four strategies will be represented by six coordinates of zeros and ones using a binary scheme. Each digit denotes the player's response when any of the six known round outcomes occur $(\mathcal{R}, \mathcal{K}, \mathcal{S}, \mathcal{T}, \mathcal{L}, \mathcal{P})$. Ones and zeros show that the player will move D for a zero and C for a one in the future. Whereas the binary representation of S_{36} is $(1, 0, 0, 1, 0, 0)$ and known as TFT, the representation of S_{63} is $(1, 1, 1, 1, 1, 1)$ which is call All C. For example, after S_{36} has just defected D, he will remain in this D stat if at least one of the opponent just played D, but will shift to its C state if the both opponents just play C as shown in Fig. 1. For another example, after S_{35} has just defected D, he will remain in this D state if the both opponents just played C, but will shift to its C state if at least one of the opponents just played D as shown in Fig. 2.

3.1. TFT strategies with memory-two

We will introduce four different strategies S_{36}, S_{38}, S_{52} and S_{54}, which are called TFT 1, TFT 2, TFT 3 and TFT 4, respectively. These four strategies are presented by the automatons in Fig. 1, such that:

We study an example for the three players S_{36}, S_{38} and S_{52}, such as player I (S_{36}) against player III (S_{52}) with player II (S_{38}). We will get the following eight cases, and every case has eight sub-cases as follows:



(a) $S_{36}(1, 0, 0, 1, 0, 0)$ (b) $S_{38}(1, 0, 0, 1, 1, 0)$

(c) $S_{52}(1, 1, 0, 1, 0, 0)$ (d) $S_{54}(1, 1, 0, 1, 1, 0)$

Fig. 1. Automates of TFT strategies.

Case 1: in round 1, players I, II and III play C, and in round 2, we discuss all eight possible cases.
Case 2: in round 1, players I and II play C, but player III plays D and in round 2, we discuss all eight possible cases.

Case 3: in round 1, players I and III play C, but player II plays D and in round 2, we discuss all eight possible cases.
Case 4: in round 1, player I plays C, but players II and III play D and in round 2, we discuss all eight possible cases.
Case 5: in round 1, player I plays D, but players II and III play C and in round 2, we discuss all eight possible cases.
Case 6: in round 1, players I and III play D, but player II plays C and in round 2, we discuss all eight possible cases.
Case 7: in round 1, players I and II plays D, but player III plays C and in round 2, we discuss all eight possible cases.
Case 8: in round 1, players I, II and III play D, and in round 2, we discuss all eight possible cases.

- Case 1.1: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and also in state 2, the three players (S_{36}, S_{38}, S_{52}) start with C.

$$\begin{array}{llll} S_{36} & C & C & C \\ S_{38} & C & C & C \\ S_{52} & C & C & C \end{array} \overbrace{\underbrace{\begin{array}{c} C \\ C \\ C \end{array}}} \begin{array}{cccccccccl} C & C & C & C & C & C & C & C & \dots & \\ C & C & C & C & C & C & C & C & \dots & \mathcal{R} \\ C & C & C & C & C & C & C & C & \dots & \end{array}$$

- Case 1.2: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, players I and II (S_{36}, S_{38}) start with C but player III (S_{52}) starts with D.

$$\begin{array}{llllllllll} S_{36} & C & C & C & D & C & D & C & D & C \\ S_{38} & C & C & C & D & C & C & C & D & C \\ S_{52} & C & D & C & C & C & D & C & D & C \end{array} \overbrace{\underbrace{\begin{array}{c} D \\ D \\ D \end{array}}} \begin{array}{cccc} C & D & \dots & \\ C & D & \dots & \frac{\mathcal{R}+\mathcal{P}}{2} \\ C & D & \dots & \end{array}$$

- Case 1.3: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, players I and III (S_{36}, S_{52}) start with C but player II (S_{38}) starts with D.

$$\begin{array}{llllll} S_{36} & C & C & C & D & C \\ S_{38} & C & D & C & C & C \\ S_{52} & C & C & C & C & C \end{array} \overbrace{\underbrace{\begin{array}{ccc} C & C & D \\ D & C & C \\ C & C & C \end{array}}} \begin{array}{cccccc} C & C & C & D & \dots & \\ C & D & C & C & \dots & \frac{2\mathcal{R}+\mathcal{T}+\mathcal{K}}{4} \\ C & C & C & C & \dots & \end{array}$$

- Case 1.4: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, players I (S_{36}) starts with C but player II and III (S_{38}, S_{52}) start with D.

$$\begin{array}{llllllll} S_{36} & C & C & C & D & C & D & C \\ S_{38} & C & D & C & C & C & D & C \\ S_{52} & C & D & C & D & C & D & C \end{array} \overbrace{\underbrace{\begin{array}{c} D \\ D \\ D \end{array}}} \begin{array}{cccccc} C & D & C & D & \dots & \\ C & D & C & D & \dots & \frac{\mathcal{R}+\mathcal{P}}{2} \\ C & D & C & D & \dots & \end{array}$$

- Case 1.5: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, players II and III (S_{38}, S_{52}) start with C but player I (S_{38}) starts with D.

$$\begin{array}{llllll} S_{36} & C & D & C & C & C \\ S_{38} & C & C & C & D & C \\ S_{52} & C & C & C & C & C \end{array} \overbrace{\underbrace{\begin{array}{ccc} D & C & C \\ C & C & D \\ C & C & C \end{array}}} \begin{array}{cccccc} C & D & C & C & \dots & \\ C & C & C & D & \dots & \frac{2\mathcal{R}+\mathcal{T}+\mathcal{K}}{4} \\ C & C & C & C & \dots & \end{array}$$

- Case 1.6: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, players II (S_{38}) starts with C but players I and III (S_{36}, S_{52}) start with D.

$$\begin{array}{llllll} S_{36} & C & D & C & D & C \\ S_{38} & C & C & C & D & C \\ S_{52} & C & D & C & D & C \end{array} \overbrace{\underbrace{\begin{array}{c} D \\ D \\ D \end{array}}} \begin{array}{cccccccc} C & D & C & D & C & D & \dots & \\ C & D & C & D & C & D & \dots & \frac{\mathcal{R}+\mathcal{P}}{2} \\ C & D & C & D & C & D & \dots & \end{array}$$

- Case 1.7: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, player III (S_{52}) starts with C but players I and II (S_{36}, S_{38}) start with D.

$$\begin{array}{llllllll} S_{36} & C & D & C & D & C & D & C \\ S_{38} & C & D & C & C & C & D & C \\ S_{52} & C & C & C & D & C & D & C \end{array} \overbrace{\underbrace{\begin{array}{c} D \\ D \\ D \end{array}}} \begin{array}{cccccc} C & D & C & D & \dots & \\ C & D & C & D & \dots & \frac{\mathcal{R}+\mathcal{P}}{2} \\ C & D & C & D & \dots & \end{array}$$

- Case 1.8: If in state 1, the three players (S_{36}, S_{38}, S_{52}) start with C and in state 2, the three players (S_{36}, S_{38}, S_{52}) start with D.

$$\begin{array}{llll} S_{36} & C & D & C \\ S_{38} & C & D & C \\ S_{52} & C & D & C \end{array} \overbrace{\underbrace{\begin{array}{c} D \\ D \\ D \end{array}}} \begin{array}{cccccccccc} C & D & C & D & C & D & C & D & \dots & \\ C & D & C & D & C & D & C & D & \dots & \frac{\mathcal{R}+\mathcal{P}}{2} \\ C & D & C & D & C & D & C & D & \dots & \end{array}$$

From the previous case, we have three regimes $\mathcal{R}$, $\frac{\mathcal{P}+\mathcal{R}}{2}$ and $\frac{2\mathcal{R}+\mathcal{K}+\mathcal{T}}{4}$. Therefore, by use the same Technique using the algorithm in the appendix section, we get that in the cases 2, 4, 6, 7 and 8, there are three regimes $\mathcal{P}$, $\frac{\mathcal{P}+\mathcal{R}}{2}$ and $\frac{2\mathcal{P}+\mathcal{K}+\mathcal{T}}{4}$. In cases 3 and 5, also there are three regimes $\frac{2\mathcal{R}+\mathcal{K}+\mathcal{T}}{4}$, $\frac{\mathcal{T}+\mathcal{K}}{2}$ and $\frac{2\mathcal{P}+\mathcal{K}+\mathcal{T}}{4}$. Finally, we have six regimes

$$R_1 = \mathcal{R}, \tag{7}$$

$$R_2 = \frac{\mathcal{R}+\mathcal{P}}{2}, \tag{8}$$

$$R_3 = \frac{2\mathcal{R}+\mathcal{T}+\mathcal{L}}{4}, \tag{9}$$

$$R_4 = \mathcal{P}, \tag{10}$$

$$R_5 = \frac{2\mathcal{P}+\mathcal{T}+\mathcal{K}}{4}, \tag{11}$$

$$R_6 = \frac{\mathcal{T}+\mathcal{K}}{2}. \tag{12}$$

When we do the perturbation, we get the following: In regime R_1, if S_{36} and S_{38} play D instead of C, then regime R_1 will transfer to R_3, and if S_{52} plays D instead of C, then regime R_1 will transfer to R_2. In regime R_2, if S_{36} and S_{38} play D instead of C in column 1, then regime R_2 will transfer to R_5, and if S_{52} plays D instead of C in column 1, then regime R_2 will transfer to R_4. But in column 2, if S_{36}, S_{38} and S_{52} play C instead of D, then regime R_2 will not change in all these cases. In regime R_3, if S_{36} and S_{38} play D instead of C in columns 1 and 3, then regime R_3 will transfer to R_6, and if S_{52} plays D instead of C in columns 1 and 3, then regime R_3 will transfer to R_5. But in column 2, if S_{36} and S_{52} play D instead of C, then regime R_3 will transfer to R_2 and if S_{38} plays C instead of D, then regime R_3 will transfer to R_1. While in column 4, if S_{36} plays C instead of D, then regime R_3 will transfer to R_1 and if S_{38} and S_{52} play D instead of C, then regime R_3 will transfer to R_2. In regime R_4, S_{36}, S_{38} and S_{52} play C instead of D, then regime R_4 will not change in all these cases. In regime R_5, if S_{36} plays C instead of D in column 1, then regime R_5 will transfer to R_2, and if S_{38} and S_{52} play D instead of C in column 1, then regime R_5 will transfer to R_4. In columns 2 and 4, S_{36}, S_{38} and S_{52} play C instead of D, then regime R_5 will not change in all these cases. While in column 3, if S_{36} and S_{52} play D instead of C, then regime R_5 will transfer to R_4, and if S_{38} plays C instead of D, then regime R_5 will transfer to R_2. In regime R_6, if S_{36} and S_{52} play D instead of C in column 1, then regime R_6 will transfer to R_5, and if S_{38} plays C instead of D in column 1, then regime R_6 will transfer to R_3. But in column 2, if S_{36} plays C instead of D, then regime R_6 will transfer to R_3, and if S_{38} and S_{52} play D instead of C, then regime R_6 will transfer to R_5. Thus, the corresponding transition matrix becomes

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{array} \begin{array}{c} \begin{array}{cccccc} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/6 & 1/3 & 0 \\ 1/6 & 1/3 & 0 & 0 & 1/6 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/6 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 & 0 \end{array}\right] \end{array} \tag{13}$$

By calculate the left eigenvectors for the eigenvalue 1, we get the following equations:

$$-v_1 + \frac{1}{6}v_3 = 0, \tag{14}$$

$$\frac{1}{3}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 + \frac{1}{6}v_5 = 0, \tag{15}$$

$$\frac{2}{3}v_1 - v_3 + \frac{1}{3}v_6 = 0, \tag{16}$$

$$\frac{1}{6}v_2 + \frac{1}{3}v_5 = 0, \tag{17}$$

$$\frac{1}{3}v_2 + \frac{1}{6}v_3 - \frac{1}{2}v_5 + \frac{2}{3}v_6 = 0, \tag{18}$$

$$\frac{1}{3}v_3 - v_6 = 0, \tag{19}$$

By solving the linear system of the previous equations from Equations (14) to (19) with the equation $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 = 1$, then we obtain the eigenvector V as

$$V = (v_1, v_2, v_3, v_4, v_5, v_6) = (0, 0, 0, 1, 0, 0) \tag{20}$$

Now, we can get the payoff values by

$$\begin{aligned} & E(S_{36}, S_{38}, S_{52}) \\ &= v_1 R_1 + v_2 R_2 + v_3 R_3 \\ &\quad + v_4 R_4 + v_5 R_5 + v_6 R_6 \\ &= \mathcal{P} \end{aligned} \tag{21}$$

According to Equation (21), the payoff vector $\Pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ is equal to $(0, 0, 0, 0, 0, 1)$. Using the same approach depending on the algorithm in the Appendix section; we examined all possibilities for studying three strategies from the four TFT strategies, which are 64 possibilities times eight sequences. Therefore, we obtained the results shown in the following tables from Tables 1 to 4.

Table 1
The payoff for player I against player III when player II fixed with the strategy TFT 1 (S_{36})

Player II (S_{36}) is fixed	TFT 1 S_{36}	TFT 2 S_{38}	TFT 3 S_{52}	TFT 4 S_{54}
TFT 1 S_{36}	(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)
TFT 2 S_{38}	(0, 0, 0, 0, 0, 1)	(1, 3, 6, 0, 9, 5)	(0, 0, 0, 0, 0, 1)	(2, 3, 3, 3, 3, 2)
TFT 3 S_{52}	(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)
TFT 4 S_{54}	(0, 0, 0, 0, 0, 1)	(2, 6, 3, 0, 3, 2)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)

Table 2
The payoff for player I against player III when player II fixed with the strategy TFT 2 (S_{38})

Player II (S_{38}) is fixed	TFT 1 S_{36}	TFT 2 S_{38}	TFT 3 S_{52}	TFT 4 S_{54}
TFT 1 S_{36}	(0, 0, 0, 0, 0, 1)	(1, 0, 3, 3, 12, 10)	(0, 0, 0, 0, 0, 1)	(2, 3, 0, 3, 6, 2)
TFT 2 S_{38}	(0, 0, 0, 0, 0, 1)	(1, 2, 1, 1, 2, 1)	(1, 2, 1, 1, 2, 1)	(1, 0, 0, 0, 0, 0)
TFT 3 S_{52}	(0, 0, 0, 0, 0, 1)	(1, 2, 1, 1, 2, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)
TFT 4 S_{54}	(2, 6, 3, 0, 3, 2)	(0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 1)

Table 3
The payoff for player I against player III when player II fixed with the strategy TFT 3 (S_{52})

Player II (S_{52}) is fixed	TFT 1 S_{36}	TFT 2 S_{38}	TFT 3 S_{52}	TFT 4 S_{54}
TFT 1 S_{36}	(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)
TFT 2 S_{38}	(0, 0, 0, 0, 0, 1)	(1, 2, 1, 1, 2, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)
TFT 3 S_{52}	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)
TFT 4 S_{54}	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)

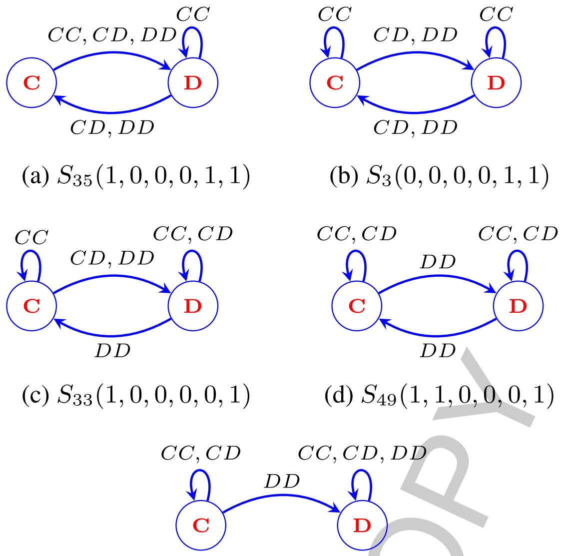
Table 4
The payoff for player I against player III when player II fixed with the strategy TFT 4 (S_{54})

Player II (S_{54}) is fixed	TFT 1 S_{36}	TFT 2 S_{38}	TFT 3 S_{52}	TFT 4 S_{54}
TFT 1 S_{36}	(0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)
TFT 2 S_{38}	(2, 3, 3, 3, 3, 2)	(5, 9, 0, 6, 3, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)
TFT 3 S_{52}	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)
TFT 4 S_{54}	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)

3.2. WSLS strategies with memory two

We will introduce five different strategies S_3, S_{35}, S_{33}, S_{49} and S_{48}, which are called WSLS 1, WSLS 2, WSLS 3, WSLS 4, and WSLS 5, respectively. These five strategies are presented by the automatons in Fig. 2, such that:

Now, we study an example for the three players S_3, S_{35} and S_{33}, such as player I (S_3) against player III (S_{33}) with player II (S_{35}). We will get the following



(a) $S_{35}(1, 0, 0, 0, 1, 1)$ (b) $S_3(0, 0, 0, 0, 1, 1)$

(c) $S_{33}(1, 0, 0, 0, 0, 1)$ (d) $S_{49}(1, 1, 0, 0, 0, 1)$

(e) $S_{48}(1, 1, 0, 0, 0, 0)$

Fig. 2. Automates of WSLS strategies.

eight cases and every case has 8 sub-cases as follows:

Case 1: in round 1, players I, II and III play C, and in round 2, we discuss all eight possible cases.
Case 2: in round 1, players I and II play C, but player III plays D and in round 2, we discuss all eight possible cases.
Case 3: in round 1, players I and III play C, but player II plays D and in round 2, we discuss all eight possible cases.

Case 4: in round 1, player I plays C, but players II and III play D and in round 2, we discuss all eight possible cases.
Case 5: in round 1, player I plays D, but players II and III play C and in round 2, we discuss all eight possible cases.

Case 6: in round 1, players I and III play D, but player II plays C and in round 2, we discuss all eight possible cases.

Case 7: in round 1, players I and II plays D, but player III plays C and in round 2, we discuss all eight possible cases.
Case 8: in round 1, players I, II and III play D, and in round 2, we discuss all eight possible cases.

- Case 1.1: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and also in state 2, the three players (S_3, S_{35}, S_{33}) start with C.

S_3	C	C	D	D	D	D	C	C	D	D	D	D	C	C	D	D	...	
S_{35}	C	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	...	$\frac{\mathcal{R}+\mathcal{T}+\mathcal{P}}{3}$
S_{33}	C	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	...	

- Case 1.2: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, players I and II (S_3, S_{35}) start with C but player III (S_{33}) starts with D.

S_3	C	C	D	D	D	C	C	D	D	D	D	C	C	D	D	D	...	
S_{35}	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	$\frac{\mathcal{R}+\mathcal{T}+\mathcal{P}}{3}$
S_{33}	C	D	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	

- Case 1.3: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, players I and III (S_3, S_{33}) start with C but player II (S_{35}) starts with D.

S_3	C	C	D	D	D	C	C	D	D	D	D	C	C	D	D	D	...	$\frac{\mathcal{R}+\mathcal{T}+\mathcal{P}}{3}$
S_{35}	C	D	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	
S_{33}	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	

- Case 1.4: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, players I (S_3) starts with C but player II and III (S_{35}, S_{33}) start with D.

S_3	C	C	D	D	D	C	C	D	D	C	D	D	C	C	D	D	...	$\frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}$
S_{35}	C	D	C	C	D	D	C	C	C	D	D	C	C	D	C	C	...	
S_{33}	C	D	C	D	D	D	C	D	C	D	D	D	C	D	C	D	...	

- Case 1.5: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, players II and III (S_{35}, S_{33}) start with C but player I (S_3) starts with D.

S_3	C	D	D	D	D	C	C	D	D	D	D	C	C	D	D	D	...	$\frac{\mathcal{R}+\mathcal{T}+\mathcal{P}}{3}$
S_{35}	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	
S_{33}	C	C	C	D	D	C	C	C	C	D	D	C	C	C	C	D	...	

- Case 1.6: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, players II (S_{35}) starts with C but players I and III (S_3, S_{33}) start with D.

S_3	C	D	D	C	D	D	C	C	D	D	D	C	C	D	D	C	...	$\frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}$
S_{35}	C	C	C	D	D	C	C	D	C	C	D	D	C	C	C	D	...	
S_{33}	C	D	C	D	D	D	C	D	C	D	D	D	C	D	C	D	...	

- Case 1.7: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, player III (S_{33}) starts with C but players I and II (S_3, S_{35}) start with D.

S_3	C	D	D	C	D	D	C	C	D	D	D	D	C	C	D	D	...	$\frac{\mathcal{R}+\mathcal{P}+\mathcal{T}}{3}$
S_{35}	C	D	C	C	D	D	C	C	C	C	D	D	C	C	C	C	...	
S_{33}	C	C	C	D	D	D	C	C	C	C	D	D	C	C	C	C	...	

- Case 1.8: If in state 1, the three players (S_3, S_{35}, S_{33}) start with C and in state 2, the three players (S_3, S_{35}, S_{33}) start with D.

S_3	C	D	D	C	D	D	C	D	D	C	D	D	C	D	D	C	...	$\frac{\mathcal{R}+\mathcal{P}+\mathcal{T}}{3}$
S_{35}	C	D	C	C	D	C	C	D	C	C	D	C	C	D	C	C	...	
S_{33}	C	D	C	C	D	C	C	D	C	C	D	C	C	D	C	C	...	

From the previous case, we have two regimes $\frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}$ and $\frac{\mathcal{R}+\mathcal{P}+\mathcal{T}}{3}$. Therefore, by using the same Technique using the algorithm in the Appendix section, we get that in cases 2, 3, 5, 7 and 8, there are two regimes $\frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}$ and $\frac{\mathcal{R}+\mathcal{P}+\mathcal{T}}{3}$. But in the cases of 4 and 6, there are two regimes $\frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}$ and $\frac{\mathcal{S}+\mathcal{L}}{2}$. Finally, we have three regimes

$$R_1 = \frac{2\mathcal{R}+2\mathcal{T}+2\mathcal{P}+3\mathcal{L}+3\mathcal{S}}{12}, \tag{22}$$

$$R_2 = \frac{\mathcal{R}+\mathcal{P}+\mathcal{T}}{3}, \tag{23}$$

$$R_3 = \frac{\mathcal{S}+\mathcal{L}}{2}, \tag{24}$$

When we do the perturbation, we get the following: In regime R_1, in columns 1 and 7, if S_3, S_{35} and S_{33} play D instead of C, then regime R_1 will not change in all these cases. In columns 2, 6 and 10, if S_3 plays D instead of C and if S_{35} and S_{33} play C instead of D then regime R_1 will transfer to R_2.

In columns 3 and 9, if S_3 plays C instead of D and if S_{35} plays D instead of C, then regime R_1 will not change, but if S_{33} plays D instead of C, then regime R_1 will transfer to R_2. In columns 4, 8 and 12, if S_3 and S_{33} play C instead of D and if S_{35} plays D instead of C, then regime R_1 in all these cases will transfer to R_2. In columns 5 and 11, if S_3 and S_{35} play C instead of D, then regime R_1 will transfer to R_2, but and if S_{33} plays C instead of D, then regime R_1 will not change. In regime R_2, in column 1, if S_3, S_{35} and S_{33} play D instead of C, then regime R_2 in all these cases will not change. But in column 2, if S_3 and S_{35} play C instead of D, then regime R_2 will transfer to R_1, and if S_{33} plays C instead of D, then regime R_2 will not change. While in column 3, if S_3 plays C instead of D and if S_{35} plays D instead of C, then regime R_2 will not change, but if S_{33} plays D instead of C, then regime R_2 will transfer to R_1. In regime R_3, in column 1, if S_3 and S_{33} play C instead of D and S_{35} play D instead of C, then regime R_3 in all these cases will transfer to R_1. Also in column 2, if S_3 plays D instead of C and if S_{35} and S_{33} play C instead of D, then regime R_3 in all these cases will transfer to R_1. Thus, the corresponding transition matrix becomes

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} 5/9 & 7/18 & 1/18 \\ 1/3 & 2/3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}. \tag{25}$$

By calculating the left eigenvectors for the eigenvalue 1, we get the following equations:

$$-\frac{4}{9}v_1 + \frac{1}{3}v_2 + v_3 = 0, \tag{26}$$

$$\frac{7}{18}v_1 - \frac{1}{3}v_2 = 0, \tag{27}$$

$$\frac{1}{18}v_1 - v_3 = 0. \tag{28}$$

By solving the linear system of the previous equations from Equations (27) to (29) with the equation $v_1 + v_2 + v_3 = 1$, then we obtain the eigenvector V as

$$V = (v_1, v_2, v_3) = \left(\frac{9}{20}, \frac{21}{40}, \frac{1}{40}\right). \tag{29}$$

Now, we can get the payoff values by

$$\begin{aligned} &E(S_{48}, S_{33}, S_{49}) \\ &= v_1.R_1 + v_2.R_2 + v_3.R_3 \\ &= \frac{20\mathcal{R} + 20\mathcal{T} + 20\mathcal{P} + 10\mathcal{L} + 10\mathcal{S}}{80} \\ &= \frac{2\mathcal{R} + 2\mathcal{T} + 2\mathcal{P} + \mathcal{L} + \mathcal{S}}{8}. \end{aligned} \tag{30}$$

According to Equation (31), the payoff vector $\Pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ is equal to $(\frac{2}{8}, 0, \frac{1}{8}, \frac{2}{8}, \frac{1}{8}, \frac{2}{8})$, but we can use

$$\begin{aligned} \Pi &= (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) \\ &= (\frac{n_1}{N}, \frac{n_2}{N}, \frac{n_3}{N}, \frac{n_4}{N}, \frac{n_5}{N}, \frac{n_6}{N}) \\ &= \frac{1}{N}(n_1, n_2, n_3, n_4, n_5, n_6), \end{aligned} \tag{31}$$

where

$$N = \sum_{i=1}^{6} n_i. \tag{32}$$

Table 5
The payoff for player I against player III when player II fixed with the strategy WSLS 1 (S_3)

Player II (S_3) is fixed	WSLS 1 S_3	WSLS 2 S_{35}	WSLS 3 S_{33}	WSLS 4 S_{49}	WSLS 5 S_{48}
WSLS 1 S_3	(1, 0, 0, 0, 0, 1)	(1, 1, 0, 0, 1, 1)	(3, 3, 2, 0, 5, 3)	(3, 3, 2, 0, 5, 3)	(0, 3, 2, 0, 2, 3)
WSLS 2 S_{35}	(1, 0, 1, 1, 0, 1)	(1, 1, 0, 0, 0, 1)	(2, 2, 1, 0, 1, 2)	(6, 12, 5, 0, 11, 6)	(0, 3, 2, 0, 2, 3)
WSLS 3 S_{33}	(3, 0, 3, 3, 4, 3)	(1, 1, 0, 0, 1, 1)	(1, 1, 0, 0, 0, 1)	(1, 1, 0, 0, 2, 1)	(0, 1, 0, 0, 0, 1)
WSLS 4 S_{49}	(3, 0, 3, 3, 4, 3)	(3, 3, 3, 3, 5, 3)	(1, 1, 1, 0, 1, 1)	(0, 1, 0, 0, 0, 0)	(0, 2, 1, 0, 1, 1)
WSLS 5 S_{48}	(0, 0, 0, 3, 4, 3)	(0, 0, 0, 3, 4, 3)	(0, 0, 0, 1, 0, 1)	(0, 1, 0, 1, 2, 1)	(0, 0, 0, 0, 1, 1)

Table 6
The payoff for player I against player III when player II fixed with the strategy WSLS 2 (S_{35})

Player II (S_{35}) is fixed	WSLS 1 S_3	WSLS 2 S_{35}	WSLS 3 S_{33}	WSLS 4 S_{49}	WSLS 5 S_{48}
WSLS 1 S_3	(1, 1, 0, 0, 1, 1)	(1, 0, 0, 1, 0, 1)	(2, 0, 1, 2, 1, 2)	(6, 6, 5, 6, 11, 6)	(0, 3, 2, 0, 2, 3)
WSLS 2 S_{35}	(1, 1, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 3, 2, 0, 2, 3)
WSLS 3 S_{33}	(1, 1, 0, 0, 1, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 3, 0, 0, 0, 3)
WSLS 4 S_{49}	(3, 3, 3, 3, 5, 3)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(3, 1, 0, 0, 0, 0)	(1, 3, 2, 0, 2, 2)
WSLS 5 S_{48}	(0, 0, 0, 3, 4, 3)	(1, 0, 0, 3, 4, 3)	(1, 0, 0, 3, 0, 3)	(1, 1, 0, 2, 4, 2)	(0, 0, 0, 0, 1, 1)

Table 7
The payoff for player I against player III when player II fixed with the strategy WSLS 3 (S_{33})

Player II (S_{33}) is fixed	WSLS 1 S_3	WSLS 2 S_{35}	WSLS 3 S_{33}	WSLS 4 S_{49}	WSLS 5 S_{48}
WSLS 1 S_3	(3, 3, 2, 0, 5, 3)	(2, 0, 1, 2, 1, 2)	(1, 0, 0, 1, 0, 1)	(1, 0, 1, 1, 1, 1)	(0, 1, 0, 0, 0, 1)
WSLS 2 S_{35}	(2, 2, 1, 0, 1, 2)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 3, 0, 0, 0, 3)
WSLS 3 S_{33}	(1, 1, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(1, 3, 0, 0, 0, 3)
WSLS 4 S_{49}	(1, 1, 1, 0, 1, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(3, 1, 0, 0, 0, 0)	(1, 3, 2, 0, 0, 2)
WSLS 5 S_{48}	(0, 0, 0, 1, 0, 1)	(1, 0, 0, 3, 0, 3)	(1, 0, 0, 3, 0, 3)	(1, 1, 0, 2, 2, 2)	(0, 0, 0, 0, 1, 1)

Table 8
The payoff for player I against player III when player II fixed with the strategy WSLS 4 (S_{49})

Player II (S_{49}) is fixed	WSLS 1 S_3	WSLS 2 S_{35}	WSLS 3 S_{33}	WSLS 4 S_{49}	WSLS 5 S_{48}
WSLS 1 S_3	(3, 3, 2, 0, 5, 3)	(6, 6, 5, 6, 11, 6)	(1, 0, 1, 1, 1, 1)	(0, 0, 0, 1, 0, 0)	(0, 1, 1, 1, 1, 1)
WSLS 2 S_{35}	(6, 12, 5, 0, 11, 5)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(3, 0, 0, 1, 0, 0)	(1, 2, 2, 1, 2, 2)
WSLS 3 S_{33}	(1, 1, 0, 0, 2, 1)	(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)	(3, 0, 0, 1, 0, 0)	(1, 2, 0, 1, 2, 2)
WSLS 4 S_{49}	(0, 1, 0, 0, 0, 0)	(3, 1, 0, 0, 0, 0)	(3, 1, 0, 0, 0, 0)	(3, 2, 0, 1, 0, 0)	(3, 2, 0, 1, 0, 0)
WSLS 5 S_{48}	(0, 1, 0, 1, 2, 1)	(1, 1, 0, 2, 4, 2)	(1, 1, 0, 2, 2, 2)	(3, 2, 0, 7, 0, 0)	(3, 5, 0, 4, 9, 9)

Table 9
The payoff for player I against player III when player II fixed with the strategy WSLS 5 (S_{48})

Player II (S_{48}) is fixed	WSLS 1 S_3	WSLS 2 S_{35}	WSLS 3 S_{33}	WSLS 4 S_{49}	WSLS 5 S_{48}
WSLS 1 S_3	(0, 3, 2, 0, 2, 3)	(0, 3, 2, 0, 2, 3)	(0, 1, 0, 0, 0, 1)	(0, 1, 1, 1, 1, 1)	(0, 0, 1, 0, 0, 1)
WSLS 2 S_{35}	(0, 3, 2, 0, 2, 3)	(1, 3, 2, 0, 2, 3)	(1, 3, 0, 0, 0, 3)	(1, 2, 2, 1, 2, 2)	(0, 0, 1, 0, 0, 1)
WSLS 3 S_{33}	(0, 1, 0, 0, 0, 1)	(1, 3, 0, 0, 0, 3)	(1, 3, 0, 0, 0, 3)	(1, 2, 0, 1, 2, 2)	(0, 1, 0, 0, 0, 1)
WSLS 4 S_{49}	(0, 2, 1, 0, 1, 1)	(1, 3, 2, 0, 2, 2)	(1, 3, 2, 0, 0, 2)	(3, 8, 0, 1, 0, 0)	(3, 8, 9, 1, 0, 9)
WSLS 5 S_{48}	(0, 0, 0, 0, 1, 1)	(0, 0, 0, 0, 1, 1)	(0, 0, 0, 0, 1, 1)	(3, 5, 0, 4, 9, 9)	(0, 0, 0, 0, 0, 1)

Finally, we can express the payoff as $(n_1, n_2, n_3, n_4, n_5, n_6) = (2, 0, 1, 2, 1, 2)$.

Using the same approach depending on the algorithm in the Appendix section, we examined all possibilities for studying three strategies from the five WSLS strategies, which are 64 possibilities times eight sequences. Therefore, we obtained the results shown in the following tables from Tables 5 to 9.

4. Results

In this section, we will discuss the domination between the Tit-For-Tat strategies and the domination between the Win Stay-Lose Shift strategies for (3P-IPD). We note that S_n is outcompeted by S_m if both $a_{nm} > a_{nn}$ and $a_{mm} > a_{nm}$, where a_{nn}, a_{nm}, a_{mn} and a_{mm} are elements of the payoff matrix. If the strategy S_n is outcompeted by S_m, we can write $S_n << S_m$.

The domination between TFT strategies according to Tables 1–4, is given as and the domination between WSLS strategies, according to Tables 5–9, is given as

Table 10
A list of strategies outcompeting S_n

		Player II: TFT 1 S_{36} fixed	Player II: TFT 2 S_{38} fixed	Player II: TFT 3 S_{52} fixed	Player II: TFT 4 S_{54} fixed
TFT 1	$S_{36} <<$	S_{38}, S_{52}, S_{54}	S_{38}, S_{52}, S_{54}	S_{38}, S_{52}, S_{54}	S_{52}, S_{54}
TFT 2	$S_{38} <<$	S_{54}	S_{54}	S_{54}	S_{54}
TFT 3	$S_{52} <<$	S_{54}	S_{54}	S_{54}	S_{54}
TFT 4	$S_{54} <<$	—	—	—	—

Table 11
A list of strategies outcompeting S_n

		Player II: WSLS 1 S_3 fixed	Player II: WSLS 2 S_{35} fixed	Player II: WSLS 3 S_{33} fixed	Player II: WSLS 4 S_{49} fixed	Player II: WSLS 5 S_{48} fixed
WSLS 1	$S_3 <<$	S_{48}	S_{49}, S_{48}	S_{33}, S_{48}	S_{33}, S_{48}	S_{48}, S_{49}
WSLS 2	$S_{35} <<$	S_{33}, S_{48}	—	—	—	S_{48}, S_{49}
WSLS 3	$S_{33} <<$	S_3	—	—	—	—
WSLS 4	$S_{49} <<$	S_{33}, S_{48}	—	—	S_{35}, S_{33}	—
WSLS 5	$S_{48} <<$	S_{33}	—	—	S_{35}, S_{33}	—

5. Conclusion

We studied the effect of memory length on the strategies and payoffs of players in a repeated three-player prisoner's dilemma game to examine the extent of its effect. This is done by studying the competition between strategies that have the same behavior (TFT and WSLS strategies) to each other. Based on Tables 1–11, and comparing these results with their similarity with memory-one in other papers [2, 17], we conclude that memory-two does not affect on the behavior of strategies in the three-player prisoner's dilemma game.

Conflicts of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

The following is the pseudo-code for our proposed parallel learning algorithm for simulating the iterated prisoner dilemma game with three players using the notion of memory.

```
FUNCTION MemG_Sim
BEGIN
INPUT: S1,S2,S3;

INITIALIZE:
    Strategy1 ← Dic2bin(S1),
    Strategy2 ← Dic2bin(S2),
    Strategy3 ← Dic2bin(S3),
    Strategies ← [S1,S2,S3],
    Prop ← 8,
    G_Index ← 25,
    Cases ← Zeros(3,G_Index),
    Results ← [ ],
    Positions ← [5,3,2;6,4,1;7,1,4;8,2,3;1,7,6;2,8,5;3,5,8;4,6,7],
    Payoffs ← ['P', 'L', 'l', 'T', 'S', 'K', 'k', 'R'],
    Payoffs ← Zeros(20,20);

FOR C ← 1: Prop
    Cases ← Play (C-1,Cases);
    FOR G ← 1:G_Index-1
        Cases ← Next (G,strategies,Cases);
    END FOR
    Mem[C] ← Cases;
END FOR

FOR First ← 1: Prop
    FOR Second ← 1: Prop
        Blocks ← Merge (Mem[First],Mem[Second]);
        output (Blocks);
        PARALELL FOR R ← 1 : 2* G_Index
            Pay[R] ← Payoff (Blocks[R]);
        END PARALELL FOR
        Results [First,Second] ← Redundant (Pay);
    END FOR
END FOR

PARALELL FOR i=1:Prop
    FOR j=1:Prop
        IF Length (Results [i,j]) mod 2 =0
            Duplicate (Results [i,j]);
        END IF
    END FOR
END PARALELL FOR

Regimes ← Unique (Results);
Output (Regimes);
N ← Count (Regimes);

FOR u ← 1:N
    Size ← Length (Regimes (u));
    FOR v ← 1:Size STEP 2
        First ← Getpos (Payoffs, Regimes [u,v]);
        Second ← Getpos (Payoffs, Regimes [u,v+1]);
        Change (First,Second);
        Change (Second,First);
    END FOR
END FOR

FOR u ← 1:N
    FOR v ← 1:N
        T[u,v]=Transition [u,v]/(3* Length (Regimes [u]));
    END FOR
END FOR

Dist_ V ← Solve (T);
RETURN Dist_ V ;
END FUNCTION MemG_Sim

PROCEDURE Change (Fpos,Spos)
FOR w ← 1:3
    Row ← Fpos;
    Col ← Positions (Spos, w);
    New ← Results (Row, col);
    M ← Mem (Regimes, New);
    Transition (u,M) ← Transition (u,M)+1 ;
END FOR
END PROCEDURE Change
```

where

- Zeros (n, m): initiate the elements of the given matrix nxm by zeros,
- Dic2bin (X): convert the input decimal X to its equivalent binary sequence of bits,
- Play (C): initialize the player's moves for each probability index C,
- Next (I): return the moves of the next round of the given game index I,
- Merge (B_1, B_2): integrate the values of the given two sequences B_1 and B_2 into one sequence,
- Redundant (B): eliminate redundant payoffs from the block B,
- Payoff (r): compute the payoff vector from the moves of round r,
- Unique (R): delete repeated strings from the input set R,

- Count (R): count the number of strings constituting the given set R,
- Length (s): evaluate the length of the string s,
- Getpos (P, R): get the position of the payoff P for the regime R,