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Relatedness in zero-determinant strategies

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Abstract. Relatedness is necessary and causal in the development of social life. Interlayer relatedness is a measure of how one player's decisions affect the decisions of other players in the game. The relatedness can be positive or negative. We had to determine how effective each strategy was under specific conditions, and how the correlation between players affected their payoffs. In this paper, we analytically study the strategies that enforce linear payoff relationships in the Iterated Prisoner's Dilemma (IPD) game considering both a relatedness factor. As a result, we first reveal that the payoffs of two players and three players can be represented by the form of determinants as shown by Press and Dyson even with the factor.

Keywords: Equalizer, iterated prisoner's dilemma (IPD), relatedness, two-player, three-player, zero-determinant strategies (ZD)

1. Introduction

Less than a decade ago, Press and Dyson discovered a new class of conditional and probabilistic tactics, called zero-determinant strategies, on two players. Zero determinant (ZD) is applied to the Iterated Prisoner's Dilemma (IPD) game through two actions, *C* or *D*. ZD tactics have been studied for two players [23] and more recently for three players [24, 30]. Interestingly, a player using ZD methods can predict the opponent's payoffs in the game, this is a great thing.

Many researchers were motivated by the discovery of Zero determinant (ZD) strategies. When Stewart and Plotkin posed the question [6], the development or origin of ZD techniques became one of the main focuses of subsequent studies [1–5]. Then, this research expanded into a variety of fields, including multiplayer games [2, 12, 21], continuous action spaces [7], alternating games [7], asymmetric games [23], animal contests [8], human reactions to computerized ZD strategies [10], and human-human experiments [11, 22], all of which help us understand the traits of human cooperation. For additional information, see the most recent elegant classification of strategies, partners (referred to as "good strategies" in References [19]), and rivals, in direct reciprocity [13]. ZD techniques have recently been used in technological fields and human interaction [9, 27, 28].

Zero determinant (ZD) strategies are a class of conditional strategies used in repeated games, especially in the context of the iterative prisoner's dilemma (IPD). These strategies are characterized by their ability to impose the desired outcome unilaterally, regardless of the opponent's actions. The distinctive feature of ZD strategies is that they limit the opponent's expected payoff to a specific value, thus limiting the opponent's ability to maximize his or her payoff. ZD strategies have received much attention in game theory and evolutionary biology due to their interesting properties and implications for cooperation and competition in repeated interactions. However, their effectiveness and stability in real-world scenarios remain a subject of ongoing research and debate. Furthermore, ZD strategies encompass two main types: equalizing strategies and extortionate strategies. The equalizer is a special case of the ZD strategies in which the focal person unilaterally chooses the payoff that the co-player receives regardless of what the co-player does, within a range of the

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co-player's payoff value [20, 25]. As a different particular example, the focus person can decide on an "extortionate" portion of the payout relative to the payoff of the co-player.

The equalizer, a specific type of ZD strategy, allows the focal player to set the co-player's payoff within a defined range, effectively equalizing the payoffs between the players. These strategies are notable for their capacity to manipulate outcomes and influence cooperation dynamics in repeated interactions, making them subjects of extensive study in game theory and evolutionary biology.

Extortionate strategies, in the context of game theory and repeated games such as the Iterated Prisoner's Dilemma (IPD), are strategies employed by a player to exploit their opponent by coercing them into unfavorable outcomes. In the case of extortionate strategies, a player aims to ensure that they receive a higher payoff than their opponent, often at the expense of the opponent's payoff. This is achieved through the use of conditional actions that manipulate the opponent's incentives to cooperate or defect, thereby maximizing the player's own payoff while minimizing the opponent's. Extortionate strategies are characterized by their ability to enforce unequal payoffs between players, typically by threatening severe consequences for non-cooperation or by offering rewards for cooperation that heavily favor the player employing the strategy [Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent].

The prisoner's dilemma game is a model of cognitive and evolutionary behavior, especially including the emergence of relationships. It is generally assumed that there is no simple ultimatum mechanism by which a player can demand unilateral action in exchange for an unfair share of the reward. Here, we show that such patterns are coincidental. In general, player versus player, the best response in player development is to accept the blackmail. Only a player who has a mental attitude towards his opponent can perform well, and in this case the prison problem that follows is a game of ultimatum.

This technical paper expands on the zero-determinant (ZD) strategies proposed by Press and Dawson for general finite games. Using the stratified linear programming (STP) Strategy, the basic formulation for designing ZD strategies in the context of general finite games is presented. In addition, STP makes the design process very easy. The rationality conditions that must be met for ZD strategies to be considered effective are presented. Sufficient conditions to ensure the effectiveness of ZD strategies designed based on STP technology are also presented. Some numerical examples were also presented to illustrate the efficiency of the method proposed in this paper [14]. Using this technique, researchers and those interested in game theory and ZD strategies can apply the ideas and concepts presented in this paper to design and analyze ZD strategies in diverse fields. This study enhances our understanding of how to design effective strategies based on STP Strategies, which contributes to the development and improvement of gaming strategies and their applications in various fields [14].

The topic of player relatedness in game theory is an important and interesting topic. This concept is usually understood by studying the interactions and relationships between players in the context of games, how the action of one player affects the actions of others, and how these players can be interconnected in a certain way. In our social life when a player plays with his father (or someone else he is related to) in a game, many effects can occur. The kinship relationship between players may have a psychological impact on players' behavior. For example, a player may feel more responsible or motivated to win when playing with his father or other family members. The presence of a kinship relationship may lead to a change in players' strategies. They may be more inclined to cooperate with each other, or competition between them may increase to win, or the parent may offer his child victory over him. This depends on the nature of the relationship between them and their goals in the game. It may allow one player to teach the other new skills or strategies in the game, increasing their interaction and bonding. Overall, this example shows how social and interpersonal relationships can play an important role in game theory and understanding player behavior. These relationships can make the game more complex and interactive by adding human and social factors to the sports or strategy game.

Relatedness in Zero Determinant (ZD) strategies refers to the degree of correlation or interdependence between the payoffs of players in a repeated game, particularly in the context of the Iterated Prisoner's Dilemma (IPD). In the realm of game theory, relatedness measures the extent to which the outcomes of one player are influenced by the actions or decisions of another player. In ZD strategies, relatedness plays a crucial role in shaping the effectiveness and stability of these strategies.

Relatedness in ZD strategies between players is an important concept in mathematical game theory. It can be used to improve team performance and make better decisions in the game. The relatedness between players in the context of zero determinant (ZD) strategies is an important element that can significantly influence the game dynamics and choice of strategies. In the context of ZD, one player can build on the other player's behavior and response in previous turns. In game theory, zero-determinant strategies are an effective means of influencing the results of recurrent encounters. By giving one player the ability to influence how their payout and that of their opponent are tied to one another, they are connected to the payoffs. Cooperation, competition, and the general dynamics of strategic relationships can all be significantly impacted by the ZD strategy choice in a variety of situations.

In ZD strategies with a relationship, A player can promote or prevent collaboration by influencing this relationship. Depending on the particular ZD strategy selected, the degree of relatedness can change. While some ZD methods may prioritize fairness and equal payoffs in order to foster cooperation, others may place a higher priority on maximizing one's payoff at the expense of the competition, which could result in defection and conflict.

High relatedness implies that the payoffs of players are closely tied to each other, leading to situations where actions taken by one player significantly impact the outcomes for both players. In such scenarios, ZD strategies may be more effective in manipulating the opponent's behavior and enforcing desired outcomes, as the actions of one player can have a direct and substantial effect on the payoffs of the other player. Conversely, low relatedness indicates a weaker correlation between the payoffs of players, resulting in more independent outcomes for each player. In this case, ZD strategies may be less effective or even unstable, as the ability to enforce desired outcomes becomes more challenging when players have greater autonomy in their decision-making. Overall, understanding the level of relatedness between players' payoffs is crucial for analyzing the dynamics of ZD strategies and their impact on cooperation and competition in repeated interactions.

2. Two-player game with ZD-strategies

The goal of this section is to ascertain players' payoffs when utilizing ZD methods for two players. We consider the symmetric two-player Iterated Prisoner's Dilemma (2P-IPD). In each round, each participant chooses an action, either C or D, which stands for cooperation and defection, respectively. Given below is the payoff matrix for (2P-IPD) [26]

$$\begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} \mathcal{R} & \mathcal{S} \\ \mathcal{T} & \mathcal{P} \end{bmatrix} \end{array}, \tag{1}$$

where

$$\mathcal{S} < \mathcal{P} < \mathcal{R} < \mathcal{T} \text{ and } \mathcal{R} > \frac{\mathcal{T} + \mathcal{S}}{2}. \tag{2}$$

The payoff matrix of a two-player Iterated Prisoner's Dilemma with relatedness is given in (3). Where a relationship is a number that expresses how much one player cares about another player's success. For two players, we'll employ the same inclusive fitness strategy as [15–17].

$$\begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} \mathcal{R}(1+\varpi) & \mathcal{S} + \mathcal{T}\varpi \\ \\ \mathcal{T} + \mathcal{S}\varpi & \mathcal{P}(1+\varpi) \end{bmatrix} \end{array}, \tag{3}$$

where ϖ is the parameter of the relatedness and $0 \leq \varpi \leq 1$.

Depending on the choice of the two players we will have four rounds. Each round leads to one of the four possible outcomes $(\mathcal{R}, \mathcal{R})$ for (C, C), $(\mathcal{S}, \mathcal{T})$ for (C, D), $(\mathcal{T}, \mathcal{S})$ for (D, C) or $(\mathcal{P}, \mathcal{P})$ for (D, D). We assume that both players use a single-memory strategy, which involves noting the outcome of the previous round and responding accordingly in the next round. We represent the strategy of player I by the probabilities $\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ and the strategy of player II by a vector $\mathbf{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$. Where Markov transition matrix for two players (μ_2) [26] is given by

$$\mu_2 = \begin{bmatrix} \Psi_1\Upsilon_1 & \Psi_1(1-\Upsilon_1) & (1-\Psi_1)\Upsilon_1 & (1-\Psi_1)(1-\Upsilon_1) \\ \Psi_2\Upsilon_3 & \Psi_2(1-\Upsilon_3) & (1-\Psi_2)\Upsilon_3 & (1-\Psi_2)(1-\Upsilon_3) \\ \Psi_3\Upsilon_2 & \Psi_3(1-\Upsilon_2) & (1-\Psi_3)\Upsilon_2 & (1-\Psi_3)(1-\Upsilon_2) \\ \Psi_4\Upsilon_4 & \Psi_4(1-\Upsilon_4) & (1-\Psi_4)\Upsilon_4 & (1-\Psi_4)(1-\Upsilon_4) \end{bmatrix}, \tag{4}$$

We assume the eigenvector of the matrix μ_2 is $\mathbf{u} = (u_1, u_2, u_3, u_4)$ which is respect to the eigenvalue 1, where

$$\mathbf{u}\,\mu_2 = \mathbf{u} \quad \equiv \quad \mathbf{u}\,(\mu_2 - I) = 0 \tag{5}$$

Let $\mu_2^* = \mu_2 - I$, then

$$\mathbf{u}\,(\mu_2 - I) = 0$$
$$\mathbf{u}\,\mu_2^* = 0 \tag{6}$$

μ_2^* is singular and represent as

$$\mu_2^* = \begin{bmatrix} \Psi_1\Upsilon_1 - 1 & \Psi_1(1-\Upsilon_1) & (1-\Psi_1)\Upsilon_1 & (1-\Psi_1)(1-\Upsilon_1) \\ \Psi_2\Upsilon_3 & \Psi_2(1-\Upsilon_3) - 1 & (1-\Psi_2)\Upsilon_3 & (1-\Psi_3)(1-\Upsilon_3) \\ \Psi_3\Upsilon_2 & \Psi_3(1-\Upsilon_2) & (1-\Psi_3)\Upsilon_2 - 1 & (1-\Psi_2)(1-\Upsilon_2) \\ \Psi_4\Upsilon_4 & \Psi_4(1-\Upsilon_4) & (1-\Psi_4)\Upsilon_4 & (1-\Psi_4)(1-\Upsilon_4) - 1 \end{bmatrix}, \tag{7}$$

Let the transposes of the second and third columns be denoted by $\mathbf{\Psi^*} = (\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4)$ and $\mathbf{\Upsilon^*} = (\Upsilon_1 - 1, \Upsilon_3, \Upsilon_2 - 1, \Upsilon_4)$, respectively. It is clear that $\mathbf{\Psi^*}$ depends only on player I 's strategy while $\mathbf{\Upsilon^*}$ depends on player II 's strategy only.

After applying some columns' elementary operations, we get

$$\mu_2^{**} =$$

$$\begin{bmatrix} \Psi_1\Upsilon_1 - 1 & \Psi_1 - 1 & \Upsilon_1 - 1 & (1-\Psi_1)(1-\Upsilon_1) \\ \Psi_2\Upsilon_3 & \Psi_2 - 1 & \Upsilon_3 & (1-\Psi_2)(1-\Upsilon_3) \\ \Psi_3\Upsilon_2 & \Psi_3 & \Upsilon_2 - 1 & (1-\Psi_3)(1-\Upsilon_2) \\ \Psi_4\Upsilon_4 & \Psi_4 & \Upsilon_4 & (1-\Psi_4)(1-\Upsilon_4) - 1 \end{bmatrix}, \tag{8}$$

$$D(\mathbf{\Psi}, \mathbf{\Upsilon}, \boldsymbol{L}) = det \begin{bmatrix} \Psi_1\Upsilon_1 - 1 & \Psi_1 - 1 & \Upsilon_1 - 1 & l_1 \\ \Psi_2\Upsilon_3 & \Psi_2 - 1 & \Upsilon_3 & l_2 \\ \Psi_3\Upsilon_2 & \Psi_3 & \Upsilon_2 - 1 & l_3 \\ \Psi_4\Upsilon_4 & \Psi_4 & \Upsilon_4 & l_4 \end{bmatrix}, \tag{9}$$

where: $\mathbf{L} = (l_1, l_2, l_3, l_4) = \big((1-\Psi_1)(1-\Upsilon_1), (1-\Psi_2)(1-\Upsilon_3), (1-\Psi_3)(1-\Upsilon_2), (1-\Psi_4)(1-\Upsilon_4) - 1\big)$.

We will now deduce the reward for each player given the relationship between the players with a factor ϖ. We define the payoff vectors for the first and second player, respectively as $\mathbf{\Gamma_1} = (\mathcal{R}(1+\varpi), \mathcal{S} + \mathcal{T}\varpi, \mathcal{T} + \mathcal{S}\varpi, \mathcal{P}(1+\varpi))$ and $\mathbf{\Gamma_2} = (\mathcal{R}(1+\varpi), \mathcal{T} + \mathcal{S}\varpi, \mathcal{S} + \mathcal{T}\varpi, \mathcal{P}(1+\varpi))$, respectively. Consequently, the anticipated payout each round for each player under infinite recurrence using the Markov chain, Pay_1 and Pay_2, are

$$Pay_1 = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Gamma_1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{1})}$$
$$Pay_2 = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Gamma_2})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{1})}, \tag{10}$$

where $\mathbf{1} = (1, 1, 1, 1)$

Since each player's predicted payout is linear relation of his payoff vector, any linear combination between payoff is true as well. Thus, using the parameters (λ, ν, κ), the predicted payoffs of the two players can be combined linearly as shown below.

$$\lambda Pay_1 + \nu Pay_2 + \kappa = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \lambda\mathbf{\Gamma_1} + \nu\mathbf{\Gamma_2} + \kappa\mathbf{1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{1})} \tag{11}$$

If two of a matrix's columns are the same or if one of its columns is a multiple of the other, then each matrix has a zero determinant value. If player I chooses the strategy $\mathbf{\Psi^*} = (\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4) = \lambda\mathbf{\Gamma_1} + \nu\mathbf{\Gamma_2} + \kappa\mathbf{1}$, then the determinant value will no longer exist because the second column will be the same as the fourth column. Similarly, if $\mathbf{\Upsilon^*} = (\Upsilon_1 - 1, \Upsilon_3, \Upsilon_2 - 1, \Upsilon_4) = \lambda\mathbf{\Gamma_1} + \nu\mathbf{\Gamma_2} + \kappa\mathbf{1}$. Therefore,

$$\lambda Pay_1 + \nu Pay_2 + \kappa = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \lambda\mathbf{\Gamma_1} + \nu\mathbf{\Gamma_2} + \kappa\mathbf{1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{1})} = 0 \tag{12}$$

There are particular types of ZD strategies, including **Equalizer** ZD strategies that let a player change the expected reward for his opponent unilaterally. We will now study this kind in the presence of a relationship between players by setting $\lambda = 0$ in (12), then $\nu Pay_2 + \kappa = 0$ (i.e. $Pay_2 = -\frac{\kappa}{\nu}$) and $\mathbf{\Psi^*} = \nu\mathbf{\Gamma_2} + \kappa\mathbf{1}$. So

$$\mathbf{\Psi^*} = \nu\mathbf{\Gamma_2} + \kappa\mathbf{1}$$
$$(\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4) = \nu\mathbf{\Gamma_2} + \kappa\mathbf{1}$$
$$(\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4) = \nu(\mathcal{R}(1+\varpi), \mathcal{T} + \mathcal{S}\varpi, \mathcal{S} + \mathcal{T}\varpi, \mathcal{P}(1+\varpi)) + \kappa(1, 1, 1, 1) \tag{13}$$

Then,

$$\Psi_1 - 1 = \nu(\mathcal{R}(1+\varpi)) + \kappa \tag{14}$$
$$\Psi_2 - 1 = \nu(\mathcal{T} + \mathcal{S}\varpi) + \kappa \tag{15}$$
$$\Psi_3 = \nu(\mathcal{S} + \mathcal{T}\varpi) + \kappa \tag{16}$$
$$\Psi_4 = \nu(\mathcal{P}(1+\varpi)) + \kappa \tag{17}$$

So,

$$\nu = \frac{(\Psi_1 - 1) - \Psi_4}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \tag{18}$$

$$\kappa = \frac{(1-\Psi_1)\mathcal{P}(1+\varpi) + \Psi_4\mathcal{R}(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)} = \frac{(1-\Psi_1)\mathcal{P} + \Psi_4\mathcal{R}}{(\mathcal{R} - \mathcal{P})} \tag{19}$$

From (15) and (16)

$$\Psi_2 = \frac{(\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \tag{20}$$

$$\Psi_3 = \frac{(\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)] + \Psi_4[\mathcal{R}(1+\varpi) - (\mathcal{S} + \mathcal{T}\varpi)]}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \tag{21}$$

Therefore, Player II 's payoff will be

$$Pay_2 = -\frac{\kappa}{\nu} = -\frac{[(1-\Psi_1)\mathcal{P} + \Psi_4\mathcal{R}](1+\varpi)}{(\Psi_1 - 1) - \Psi_4} = \frac{[(1-\Psi_1)\mathcal{P} + \Psi_4\mathcal{R}](1+\varpi)}{\Psi_4 - (\Psi_1 - 1)} \tag{22}$$

Furthermore, player I's strategy become

$$\mathbf{\Psi^*} = \Bigg(\Psi_1, \frac{(\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)}, \frac{(\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{R}(1+\varpi)]}{(\mathcal{R} - \mathcal{P})(1+\varpi)}, \Psi_4\Bigg) \tag{23}$$

Since the values of $\Psi_i \in [0, 1]$ where $i = 1, 2, 3$ or 4, then the values of Ψ_1 and Ψ_4 must be

$$\Psi_1 \in \left[\max\left\{1 - \frac{(\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)}\ ,\ 1 - \frac{(\mathcal{R} - \mathcal{P})(1+\varpi)}{\mathcal{P}(1+\varpi) - (\mathcal{S} + \mathcal{T}\varpi)}\right\}, 1\right] \tag{24}$$

$$\Psi_4 \in \left[0, \min\left\{\frac{(\mathcal{R} - \mathcal{P})(1+\varpi) + (\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)]}{[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)]}\ ,\ \frac{(\mathcal{R} - \mathcal{P})(1+\varpi) - (\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)]}{[\mathcal{R}(1+\varpi) - (\mathcal{S} + \mathcal{T}\varpi)]}\right\}\right] \tag{25}$$

Similar for get Pay_1, we put $\nu = 0$

$$\mathbf{\Psi^*} = \lambda\mathbf{\Gamma_1} + \kappa\mathbf{1}$$
$$(\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4) = \lambda\mathbf{\Gamma_1} + \kappa\mathbf{1}$$
$$(\Psi_1 - 1, \Psi_2 - 1, \Psi_3, \Psi_4) = \lambda(\mathcal{R}(1+\varpi), \mathcal{S} + \mathcal{T}\varpi, \mathcal{T} + \mathcal{S}\varpi, \mathcal{P}(1+\varpi)) + \kappa(1, 1, 1, 1) \tag{26}$$

Then,

$$\Psi_1 - 1 = \lambda(\mathcal{R}(1+\varpi)) + \kappa \tag{27}$$
$$\Psi_2 - 1 = \lambda(\mathcal{S} + \mathcal{T}\varpi) + \kappa \tag{28}$$
$$\Psi_3 = \lambda(\mathcal{T} + \mathcal{S}\varpi) + \kappa \tag{29}$$
$$\Psi_4 = \lambda(\mathcal{P}(1+\varpi)) + \kappa \tag{30}$$

So,

$$\lambda = \frac{(\Psi_1 - 1) - \Psi_4}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \tag{31}$$

$$\kappa = \frac{(1-\Psi_1)\mathcal{P}(1+\varpi) + \Psi_4\mathcal{R}(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)} = \frac{(1-\Psi_1)\mathcal{P} + \Psi_4 R}{(\mathcal{R} - \mathcal{P})} \tag{32}$$

Table 1
Payoff with Strong Relatedness $\varpi = 0.9$

So, player I's payoff will be

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
S_0	1.9	3.45	1.9	3.45	-	5	-	5	1.9	3.45	1.9	3.45	-	5	-	5
S_1	3.2	3.8	3.8	3.8	-	-	5	-	3.2	4.2	3.8	4.2	-	5	5	5
S_2	1.9	3.8	-	4.75	1.9	4.2	1.9	4.2	1.9	3.8	-	4.75	-	5.35	-	5.35
S_3	3.2	3.8	4.75	-	3.8	3.8	4.275	3.8	3.2	4.275	4.75	-	-	5.35	5.35	5.35
S_4	-	-	1.9	3.45	-	-	-	5	-	-	1.9	3.45	-	-	-	5
S_5	4.5	-	4.033	3.8	-	-	-	-	4.5	-	4.275	4.2	-	-	5	5
S_6	-	4.5	1.9	4.275	-	-	1.9	4.2	-	4.5	-	5.067	-	-	-	5.35
S_7	4.5	-	4.033	3.8	4.5	-	4.033	3.8	4.5	4.5	5.067	5.067	-	-	5.35	5.35
S_8	1.9	3.45	1.9	3.45	-	5	-	5	-	-	-	-	-	-	-	-
S_9	3.2	4.033	3.8	4.275	-	-	5	5	-	5.7	-	5.7	-	-	-	-
S_{10}	1.9	3.8	-	4.75	1.9	4.275	-	5.067	-	-	-	-	-	5.7	-	5.7
S_{11}	3.2	4.033	4.75	4.75	3.2	4.033	5.067	5.067	-	5.7	-	-	-	5.7	5.7	5.7
S_{12}	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
S_{13}	4.5	4.5	5.1	5.1	-	-	-	-	-	-	5.7	5.7	-	-	-	-
S_{14}	-	4.5	-	5.1	-	4.5	-	5.1	-	-	-	5.7	-	-	-	5.7
S_{15}	4.5	4.5	5.1	5.1	4.5	4.5	5.1	5.1	-	-	5.7	5.7	-	-	5.7	5.7

From (28) and (29)

$$\Psi_2 = \frac{(\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)] + \Psi_4[\mathcal{R}(1+\varpi) - (\mathcal{S} + \mathcal{T}\varpi)] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (33)$$

$$\Psi_3 = \frac{(\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)]}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (34)$$

Table 2
Payoff with Weak Relatedness $\varpi = 0.1$

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
S_0	1.1	3.05	1.1	3.05	-	5	-	5	1.1	3.05	1.1	3.05	-	5	-	5
S_1	0.8	2.2	2.2	2.2	-	-	5	-	0.8	3.133	2.2	3.133	-	5	5	5
S_2	1.1	2.2	-	2.75	1.1	3.133	1.1	3.133	1.1	2.2	-	2.75	-	4.15	-	4.15
S_3	0.8	2.2	2.75	-	2.2	2.2	2.475	2.2	0.8	2.475	2.75	-	-	4.15	4.15	4.15
S_4	-	-	1.1	3.05	-	-	-	5	-	-	1.1	3.05	-	-	-	5
S_5	0.5	-	1.633	2.2	-	-	-	-	0.5	-	2.475	3.133	-	-	5	5
S_6	-	0.5	1.1	2.475	-	-	1.1	3.133	-	0.5	-	2.933	-	-	-	4.15
S_7	0.5	-	1.633	2.2	0.5	-	1.633	2.2	0.5	0.5	2.933	2.933	-	-	4.15	4.15
S_8	1.1	3.05	1.1	3.05	-	5	-	5	-	-	-	-	-	-	-	-
S_9	0.8	1.633	2.2	2.475	-	-	5	5	-	3.3	-	3.3	-	-	-	-
S_{10}	1.1	2.2	-	2.75	1.1	2.475	-	2.933	-	-	-	-	-	3.3	-	3.3
S_{11}	0.8	1.633	2.75	2.75	0.8	1.633	2.933	2.933	-	3.3	-	-	-	3.3	3.3	3.3
S_{12}	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
S_{13}	0.5	0.5	1.9	1.9	-	-	-	-	-	-	3.3	3.3	-	-	-	-
S_{14}	-	0.5	-	1.9	-	0.5	-	1.9	-	-	-	3.3	-	-	-	3.3
S_{15}	0.5	0.5	1.9	1.9	0.5	0.5	1.9	1.9	-	-	3.3	3.3	-	-	3.3	3.3

$$
\begin{aligned}
Pay_1 &= -\frac{\kappa}{\lambda} \\
&= -\frac{(1-\Psi_1)\mathcal{P}(1+\varpi) + \Psi_4\mathcal{R}(1+\varpi)}{(\Psi_1 - 1) - \Psi_4} \\
&= -\frac{[(1-\Psi_1)\mathcal{P} + \Psi_4\mathcal{R}](1+\varpi)}{(\Psi_1 - 1) - \Psi_4} \\
&= \frac{[(1-\Psi_1)\mathcal{P} + \Psi_4\mathcal{R}](1+\varpi)}{\Psi_4 - (\Psi_1 - 1)}
\end{aligned}
\tag{35}
$$

Furthermore, player I's strategy become

$$\mathbf{\Psi^*} = \Bigg(\Psi_1, \frac{(\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{R}(1+\varpi)] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)},$$
$$\frac{(\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)] - \Psi_4[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)]}{(\mathcal{R} - \mathcal{P})(1+\varpi)}, \Psi_4\Bigg) \tag{36}$$

Since the values of $\Psi_i \in [0, 1]$ where $i = 1, 2, 3$ or 4, then the values of Ψ_1 and Ψ_4 must be

$$\Psi_1 \in \left[1 - \frac{(\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)}, 1\right] \tag{37}$$

$$\Psi_4 \in \left[0, \min\left\{\frac{(\Psi_1 - 1)[(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{P}(1+\varpi)]}{(\mathcal{S} + \mathcal{T}\varpi) - \mathcal{R}(1+\varpi)}, \frac{(\Psi_1 - 1)[(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{P}(1+\varpi)]}{(\mathcal{T} + \mathcal{S}\varpi) - \mathcal{R}(1+\varpi)}\right\}\right] \tag{38}$$

We study **Equalizer** strategies at some numerical values of payoff vectors (Axelrod's values) $\mathcal{S} = 0, \mathcal{P} = 1, \mathcal{R} = 3$ and $\mathcal{T} = 5$ to expect payoff for player I against player II with a relationship between the two players on tables 1 and 2. Based on conditions (37) and (38), not all strategies represent ZD strategies Such as the strategies shown in front of them (-) in the following tables 1 and 2. Some strategies that are only based on these conditions for which we can calculate the payoff. For example, if the competition between player I (TFT (S_{10})) with player II (All D (S_0)) with ”Strong Relatedness $\varpi = 0.9$" then payoff of Player I (TFT (S_{10})) is 1.9. But with ”Weak Relatedness $\varpi = 0.1$", the payoff of Player I (TFT (S_{10})) is 1.1.

3. Three-player game with ZD-strategies

In this section, the focus is on determining the payoff of players when using ZD strategies for three players. We also consider the symmetric Iterated Prisoner's Dilemma but for three-player (3P-IPD), where each player chooses a decision from the two actions C and D. So, the payoff matrix for (3P-IPD) [29] is given by

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} CC \;\; CD \;\; DD \\ \begin{bmatrix} \mathcal{R} & \mathcal{K} & \mathcal{S} \\ \mathcal{T} & \mathcal{L} & \mathcal{P} \end{bmatrix} \end{array}, \tag{39}$$

where

$$\mathcal{S} < \mathcal{P} < \mathcal{K} < \mathcal{L} < \mathcal{R} < \mathcal{T}. \tag{40}$$

Therefore, the payoff matrix of the three-player Prisoner's Dilemma with relatedness [29] will be given by

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \quad CC \qquad\qquad CD \qquad\qquad DD \\ \begin{bmatrix} \mathcal{R}(1+\varpi) & \frac{2\mathcal{K}(1+\varpi) + \mathcal{T}\varpi}{2} & \mathcal{S} + \mathcal{T}\varpi \\ \frac{2\mathcal{T}(1+\varpi) + \mathcal{K}\varpi}{2} & \mathcal{L} + \mathcal{S}\varpi & \mathcal{P}(1+\varpi) \end{bmatrix} \end{array}. \tag{41}$$

where ϖ is the parameter of the relatedness and $0 \leq \varpi \leq 1$.

Depending on the choice of the three players we will have eight rounds which may be reduced to only six rounds. Each round leads to one of the six possible outcomes $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ for $(C, (C, C))$, $(\mathcal{K}, \mathcal{K}, \mathcal{T})$ for $(C, (C, D))$, $(\mathcal{S}, \mathcal{L}, \mathcal{L})$ for $(C, (D, D))$, $(\mathcal{T}, \mathcal{K}, \mathcal{K})$ for $(D, (C, C))$, $(\mathcal{L}, \mathcal{S}, \mathcal{L})$ for $(D, (C, D))$ or $(\mathcal{P}, \mathcal{P}, \mathcal{P})$ for $(D, (D, D))$ [18]. We postulate that each player employs the memory-one strategy, where each player observes one of the six possible outcomes in one round, and responds accordingly in the following round. We represent strategy of player I by the probabilities $\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)$, the strategy of player II by a vector $\mathbf{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5, \Upsilon_6)$ and the strategy of player II by a vector $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$. Where Markov transition matrix for three players (μ_3) [29] is given by

$$\mu_3 = \left(\begin{array}{ll} \Psi_1\Upsilon_1\Omega_1 & \Psi_1[\Upsilon_1(1-\Omega_1) + (1-\Upsilon_1)\Omega_1] \\ \Psi_2\Upsilon_2\Omega_4 & \Psi_2[\Upsilon_2(1-\Omega_4) + (1-\Upsilon_2)\Omega_4] \\ \Psi_3\Upsilon_5\Omega_5 & \Psi_3[\Upsilon_5(1-\Omega_5) + (1-\Upsilon_5)\Omega_5] \\ \Psi_4\Upsilon_2\Omega_2 & \Psi_4[\Upsilon_2(1-\Omega_2) + (1-\Upsilon_2)\Omega_2] \\ \Psi_5\Upsilon_3\Omega_5 & \Psi_5[\Upsilon_3(1-\Omega_5) + (1-\Upsilon_3)\Omega_5] \\ \Psi_6\Upsilon_6\Omega_6 & \Psi_6[\Upsilon_6(1-\Omega_6) + (1-\Upsilon_6)\Omega_6] \end{array}\right.$$

$$\begin{array}{ll} \Psi_1(1-\Upsilon_1)(1-\Omega_1) & (1-\Psi_1)\Upsilon_1\Omega_1 \\ \Psi_2(1-\Upsilon_2)(1-\Omega_4) & (1-\Psi_2)\Upsilon_2\Omega_4 \\ \Psi_3(1-\Upsilon_5)(1-\Omega_5) & (1-\Psi_3)\Upsilon_5\Omega_5 \\ \Psi_4(1-\Upsilon_2)(1-\Omega_2) & (1-\Psi_4)\Upsilon_2\Omega_2 \\ \Psi_5(1-\Upsilon_3)(1-\Omega_5) & (1-\Psi_5)\Upsilon_3\Omega_5 \\ \Psi_6(1-\Upsilon_6)(1-\Omega_6) & (1-\Psi_6)\Upsilon_6\Omega_6 \end{array}$$

$$\left.\begin{matrix}
(1-\Psi_1)[\Upsilon_1(1-\Omega_1)+(1-\Upsilon_1)\Omega_1] & (1-\Psi_1)(1-\Upsilon_1)(1-\Omega_1) \\
(1-\Psi_2)[\Upsilon_2(1-\Omega_4)+(1-\Upsilon_2)\Omega_4] & (1-\Psi_2)(1-\Upsilon_2)(1-\Omega_4) \\
(1-\Psi_3)[\Upsilon_5(1-\Omega_5)+(1-\Upsilon_5)\Omega_5] & (1-\Psi_3)(1-\Upsilon_5)(1-\Omega_5) \\
(1-\Psi_4)[\Upsilon_2(1-\Omega_2)+(1-\Upsilon_2)\Omega_2] & (1-\Psi_4)(1-\Upsilon_2)(1-\Omega_2) \\
(1-\Psi_5)[\Upsilon_3(1-\Omega_5)+(1-\Upsilon_3)\Omega_5] & (1-\Psi_5)(1-\Upsilon_3)(1-\Omega_5) \\
(1-\Psi_6)[\Upsilon_6(1-\Omega_6)+(1-\Upsilon_6)\Omega_6] & (1-\Psi_6)(1-\Upsilon_6)(1-\Omega_6)
\end{matrix}\right). \qquad (42)$$

We assume the vector $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6)$ to represent the left eigenvector of the matrix μ_3 for the eigenvalue 1. This vector represents the constant distribution of outcomes when the game is repeated infinitely where

$$\mathbf{u}\,\mu_3 = \mathbf{u} \quad \equiv \quad \mathbf{u}\,(\mu_3 - I) = 0 \qquad (43)$$

Let $\mu_3^* = \mu_3 - I$, then

$$\mathbf{u}\,(\mu_3 - I) = 0$$
$$\mathbf{u}\,\mu_3^* = 0 \qquad (44)$$

μ_3^* is singular and represent as

$$\mu_3^* = \left(\begin{matrix}
\Psi_1\Upsilon_1\Omega_1 - 1 & \Psi_1[\Upsilon_1(1-\Omega_1)+(1-\Upsilon_1)\Omega_1] & \Psi_1(1-\Upsilon_1)(1-\Omega_1) \\
\Psi_2\Upsilon_2\Omega_4 & \Psi_2[\Upsilon_2(1-\Omega_4)+(1-\Upsilon_2)\Omega_4] - 1 & \Psi_2(1-\Upsilon_2)(1-\Omega_4) \\
\Psi_3\Upsilon_5\Omega_5 & \Psi_3[\Upsilon_5(1-\Omega_5)+(1-\Upsilon_5)\Omega_5] & \Psi_3(1-\Upsilon_5)(1-\Omega_5) - 1 \\
\Psi_4\Upsilon_2\Omega_2 & \Psi_4[\Upsilon_2(1-\Omega_2)+(1-\Upsilon_2)\Omega_2] & \Psi_4(1-\Upsilon_2)(1-\Omega_2) \\
\Psi_5\Upsilon_3\Omega_5 & \Psi_5[\Upsilon_3(1-\Omega_5)+(1-\Upsilon_3)\Omega_5] & \Psi_5(1-\Upsilon_3)(1-\Omega_5) \\
\Psi_6\Upsilon_6\Omega_6 & \Psi_6[\Upsilon_6(1-\Omega_6)+(1-\Upsilon_6)\Omega_6] & \Psi_6(1-\Upsilon_6)(1-\Omega_6)
\end{matrix}\right.$$

$$\left.\begin{matrix}
(1-\Psi_1)\Upsilon_1\Omega_1 & (1-\Psi_1)[\Upsilon_1(1-\Omega_1)+(1-\Upsilon_1)\Omega_1] & (1-\Psi_1)(1-\Upsilon_1)(1-\Omega_1) \\
(1-\Psi_2)\Upsilon_2\Omega_4 & (1-\Psi_2)[\Upsilon_2(1-\Omega_4)+(1-\Upsilon_2)\Omega_4] & (1-\Psi_2)(1-\Upsilon_2)(1-\Omega_4) \\
(1-\Psi_3)\Upsilon_5\Omega_5 & (1-\Psi_3)[\Upsilon_5(1-\Omega_5)+(1-\Upsilon_5)\Omega_5] & (1-\Psi_3)(1-\Upsilon_5)(1-\Omega_5) \\
(1-\Psi_4)\Upsilon_2\Omega_2 - 1 & (1-\Psi_4)[\Upsilon_2(1-\Omega_2)+(1-\Upsilon_2)\Omega_2] & (1-\Psi_4)(1-\Upsilon_2)(1-\Omega_2) \\
(1-\Psi_5)\Upsilon_3\Omega_5 & (1-\Psi_5)[\Upsilon_3(1-\Omega_5)+(1-\Upsilon_3)\Omega_5] - 1 & (1-\Psi_5)(1-\Upsilon_3)(1-\Omega_5) \\
(1-\Psi_6)\Upsilon_6\Omega_6 & (1-\Psi_6)[\Upsilon_6(1-\Omega_6)+(1-\Upsilon_6)\Omega_6] & (1-\Psi_6)(1-\Upsilon_6)(1-\Omega_6) - 1
\end{matrix}\right). \qquad (45)$$

Let the transposes of the second and fifth columns be denoted by $\mathbf{\Delta_1^{2,3}} = (\Psi_1 - 1, \Psi_2 - 1, \Psi_3 - 1, \Psi_4, \Psi_5, \Psi_6)$ and $\mathbf{\Delta_{2,3}^{1}} = (\Upsilon_1 + \Omega_1 - 2, \Upsilon_2 + \Omega_4 - 1, \Upsilon_5 + \Omega_5, \Upsilon_2 + \Omega_2 - 2, \Upsilon_3 + \Omega_5 - 1, \Upsilon_6 + \Omega_6)$, respectively. It is clear that $\mathbf{\Delta_1^{2,3}}$ is rely only on player I 's strategy while $\mathbf{\Delta_{2,3}^{1}}$ is rely only on player II and III 's strategies.

After applying some columns' elementary operations, we get

$$\mu_3^{**} = \left(\begin{matrix}
\Psi_1\Upsilon_1\Omega_1 - 1 & \Psi_1[\Upsilon_1(1-\Omega_1)+(1-\Upsilon_1)\Omega_1] & \Psi_1 - 1 \\
\Psi_2\Upsilon_2\Omega_4 & \Psi_2[\Upsilon_2(1-\Omega_4)+(1-\Upsilon_2)\Omega_4] - 1 & \Psi_2 - 1 \\
\Psi_3\Upsilon_5\Omega_5 & \Psi_3[\Upsilon_5(1-\Omega_5)+(1-\Upsilon_5)\Omega_5] & \Psi_3 - 1 \\
\Psi_4\Upsilon_2\Omega_2 & \Psi_4[\Upsilon_2(1-\Omega_2)+(1-\Upsilon_2)\Omega_2] & \Psi_4 \\
\Psi_5\Upsilon_3\Omega_5 & \Psi_5[\Upsilon_3(1-\Omega_5)+(1-\Upsilon_3)\Omega_5] & \Psi_5 \\
\Psi_6\Upsilon_6\Omega_6 & \Psi_6[\Upsilon_6(1-\Omega_6)+(1-\Upsilon_6)\Omega_6] & \Psi_6
\end{matrix}\right.$$

$$\left.\begin{matrix}
(1-\Psi_1)\Upsilon_1\Omega_1 & \Upsilon_1 + \Omega_1 - 2 & (1-\Psi_1)(1-\Upsilon_1)(1-\Omega_1) \\
(1-\Psi_2)\Upsilon_2\Omega_4 & \Upsilon_2 + \Omega_4 - 1 & (1-\Psi_2)(1-\Upsilon_2)(1-\Omega_4) \\
(1-\Psi_3)\Upsilon_5\Omega_5 & \Upsilon_5 + \Omega_5 & (1-\Psi_3)(1-\Upsilon_5)(1-\Omega_5) \\
(1-\Psi_4)\Upsilon_2\Omega_2 - 1 & \Upsilon_2 + \Omega_2 - 2 & (1-\Psi_4)(1-\Upsilon_2)(1-\Omega_2) \\
(1-\Psi_5)\Upsilon_3\Omega_5 & \Upsilon_3 + \Omega_5 - 1 & (1-\Psi_5)(1-\Upsilon_3)(1-\Omega_5) \\
(1-\Psi_6)\Upsilon_6\Omega_6 & \Upsilon_6 + \Omega_6 & (1-\Psi_6)(1-\Upsilon_6)(1-\Omega_6) - 1
\end{matrix}\right). \qquad (46)$$

$$D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \boldsymbol{L}) = det$$

$$\left(\begin{matrix}
\Psi_1\Upsilon_1\Omega_1 - 1 & \Psi_1[\Upsilon_1(1-\Omega_1)+(1-\Upsilon_1)\Omega_1] & \Psi_1 - 1 \\
\Psi_2\Upsilon_2\Omega_4 & \Psi_2[\Upsilon_2(1-\Omega_4)+(1-\Upsilon_2)\Omega_4] - 1 & \Psi_2 - 1 \\
\Psi_3\Upsilon_5\Omega_5 & \Psi_3[\Upsilon_5(1-\Omega_5)+(1-\Upsilon_5)\Omega_5] & \Psi_3 - 1 \\
\Psi_4\Upsilon_2\Omega_2 & \Psi_4[\Upsilon_2(1-\Omega_2)+(1-\Upsilon_2)\Omega_2] & \Psi_4 \\
\Psi_5\Upsilon_3\Omega_5 & \Psi_5[\Upsilon_3(1-\Omega_5)+(1-\Upsilon_3)\Omega_5] & \Psi_5 \\
\Psi_6\Upsilon_6\Omega_6 & \Psi_6[\Upsilon_6(1-\Omega_6)+(1-\Upsilon_6)\Omega_6] & \Psi_6
\end{matrix}\right.$$

$$\left.\begin{matrix}
(1-\Psi_1)\Upsilon_1\Omega_1 & \Upsilon_1 + \Omega_1 - 2 & l_1 \\
(1-\Psi_2)\Upsilon_2\Omega_4 & \Upsilon_2 + \Omega_4 - 1 & l_2 \\
(1-\Psi_3)\Upsilon_5\Omega_5 & \Upsilon_5 + \Omega_5 & l_3 \\
(1-\Psi_4)\Upsilon_2\Omega_2 - 1 & \Upsilon_2 + \Omega_2 - 2 & l_4 \\
(1-\Psi_5)\Upsilon_3\Omega_5 & \Upsilon_3 + \Omega_5 - 1 & l_5 \\
(1-\Psi_6)\Upsilon_6\Omega_6 & \Upsilon_6 + \Omega_6 & l_6
\end{matrix}\right), \qquad (47)$$

where: $\mathbf{L} = (l_1, l_2, l_3, l_4, l_5, l_6) = ((1-\Psi_1)(1-\Upsilon_1)(1-\Omega_1), (1-\Psi_2)(1-\Upsilon_2)(1-\Omega_4), (1-\Psi_3)(1-\Upsilon_5)(1-\Omega_5), (1-\Psi_4)(1-\Upsilon_2)(1-\Omega_2), (1-\Psi_5)(1-\Upsilon_3)(1-\Omega_5), (1-\Psi_6)(1-\Upsilon_6)(1-\Omega_6)-1)$.

We will now deduce the reward for each player given the relationship between the players with a factor ϖ. We denoted the payoff vectors corresponding to player I and player II by $\mathbf{\Gamma_1} = \Big(\mathcal{R}(1+\varpi), \frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2}, \mathcal{S}+\mathcal{T}\varpi, \frac{2\mathcal{T}(1+\varpi)+\mathcal{K}\varpi}{2}, \mathcal{L}+\mathcal{S}\varpi, \mathcal{P}(1+\varpi)\Big)$. Consequently, the anticipated payout of each round for each player under infinite recurrence using the Markov chain, Pay_1, Pay_2 and Pay_3, are

$$Pay_1 = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{\Gamma_1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{1})}$$
$$Pay_2 = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{\Gamma_2})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{1})},$$
$$Pay_3 = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{\Gamma_3})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{1})}, \tag{48}$$

where $\mathbf{1} = (1, 1, 1, 1, 1, 1)$

Since each player's predicted payout depends linearly on his payoff vector, this is true for any linear combination of payoffs as well. Thus, using the parameters $(\lambda, \eta, \xi, \kappa)$, the predicted payoffs of the three players can be combined linearly as shown below.

$$\lambda Pay_1 + \eta Pay_2 + \xi Pay_3 + \kappa = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \lambda\mathbf{\Gamma_1} + \eta\mathbf{\Gamma_2} + \xi\mathbf{\Gamma_3} + \kappa\mathbf{1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{1})} \tag{49}$$

Since each matrix has a zero determinant value if there are two identical columns or if one of its columns is a multiple of the other. If player I chooses strategy $\mathbf{\Delta_1^{2,3}} = (\Psi_1 - 1, \Psi_2 - 1, \Psi_3 - 1, \Psi_4, \Psi_5, \Psi_6) = \lambda\mathbf{\Gamma_1} + \eta\mathbf{\Gamma_2} + \xi\mathbf{\Gamma_3} + \kappa\mathbf{1}$, then the determinant value will no longer exist because the third and sixth columns will be identical. Similarly, if $\mathbf{\Delta^1_{2,3}} = (\Upsilon_1 + \Omega_1 - 2, \Upsilon_2 + \Omega_4 - 1, \Upsilon_5 + \Omega_5, \Upsilon_2 + \Omega_2 - 2, \Upsilon_3 + \Omega_5 - 1, \Upsilon_6 + \Omega_6) = \lambda\mathbf{\Gamma_1} + \eta\mathbf{\Gamma_2} + \xi\mathbf{\Gamma_3} + \kappa\mathbf{1}$. Therefore,

$$\lambda Pay_1 + \eta Pay_2 + \xi Pay_3 + \kappa = \frac{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \lambda\mathbf{\Gamma_1} + \eta\mathbf{\Gamma_2} + \xi\mathbf{\Gamma_3} + \kappa\mathbf{1})}{D(\mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Omega}, \mathbf{1})} = 0 \tag{50}$$

There are particular types of ZD strategies, including **Equalizer** ZD strategies that let a player change his opponent's expected payoff unilaterally. We will now study this kind in the presence of a relationship between players by setting $\lambda = 0$ and $\eta = \xi$ in (44), then $\eta Pay_2 + \xi Pay_3 + \kappa = \eta(Pay_2 + Pay_3) + \kappa = 0$ (i.e. $Pay_2 + Pay_3 = \frac{-\kappa}{\eta}$) and $\mathbf{\Delta_1^{2,3}} = \eta(\mathbf{\Gamma_2} + \mathbf{\Gamma_3}) + \kappa\mathbf{1}$. So

$$\begin{aligned}\mathbf{\Delta_1^{2,3}} &= \eta\mathbf{\Gamma_2} + \eta\mathbf{\Gamma_3} + \kappa\mathbf{1}\\ (\Psi_1 - 1, \Psi_2 - 1, \Psi_3 - 1, \Psi_4, \Psi_5, \Psi_6) &= \eta\mathbf{\Gamma_2} + \eta\mathbf{\Gamma_3} + \kappa\mathbf{1}\\ (\Psi_1 - 1, \Psi_2 - 1, \Psi_3 - 1, \Psi_4, \Psi_5, \Psi_6) &= \eta\,[(\mathcal{R}(1+\varpi), \frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2}, \mathcal{L}+\mathcal{S}\varpi, \frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2},\\ &\quad \mathcal{S}+\mathcal{T}\varpi, \mathcal{P}(1+\varpi)) + (\mathcal{R}(1+\varpi), \frac{2\mathcal{T}(1+\varpi)+\mathcal{K}\varpi}{2}, \mathcal{L}+\mathcal{S}\varpi, \frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2},\\ &\quad \mathcal{L}+\mathcal{S}\varpi, \mathcal{P}(1+\varpi))] + \kappa(1, 1, 1, 1, 1, 1)\end{aligned} \tag{51}$$

Then,

$$\Psi_1 - 1 = 2\eta(\mathcal{R}(1+\varpi)) + \kappa \tag{52}$$

$$\Psi_2 - 1 = \eta(\frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2} + \frac{2\mathcal{T}(1+\varpi)+\mathcal{K}\varpi}{2}) + \kappa = \frac{\eta}{2}(\mathcal{T}+\mathcal{K})(3\varpi+2) + \kappa \tag{53}$$

$$\Psi_3 - 1 = 2\eta(\mathcal{L}+\mathcal{S}\varpi) + \kappa \tag{54}$$

$$\Psi_4 = \eta(2\mathcal{K}(1+\varpi) + \mathcal{T}\varpi) + \kappa \tag{55}$$

$$\begin{aligned}\mathbf{\Delta_1^{2,3}} &= \lambda\mathbf{\Gamma_1} + \kappa\mathbf{1}\\ (\Psi_1-1, \Psi_2-1, \Psi_3-1, \Psi_4, \Psi_5, \Psi_6) &= \lambda\mathbf{\Gamma_1} + \kappa\mathbf{1}\\ (\Psi_1-1, \Psi_2-1, \Psi_3-1, \Psi_4, \Psi_5, \Psi_6) &= \lambda\,(\mathcal{R}(1+\varpi),\\ &\frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2}, \mathcal{S}+\mathcal{T}\varpi, \quad \frac{2\mathcal{T}(1+\varpi)+\mathcal{K}\varpi}{2},\\ &\mathcal{L}+\mathcal{S}\varpi, \mathcal{P}(1+\varpi)) + \kappa(1,1,1,1,1,1) \qquad (65)\end{aligned}$$

$$\begin{aligned}\Psi_5 &= \eta(\mathcal{S}+\mathcal{T}\varpi+\mathcal{L}+\mathcal{S}\varpi)+\kappa\\ &= \eta(\mathcal{L}+\mathcal{S}(\varpi+1)+\mathcal{T}\varpi)+\kappa \qquad (56)\end{aligned}$$

$$\Psi_6 = 2\eta(\mathcal{P}(1+\varpi))+\kappa \qquad (57)$$

So,

$$\eta = \frac{(\Psi_1-1)-\Psi_6}{2(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (58)$$

$$\begin{aligned}\kappa &= \frac{(1-\Psi_1)\mathcal{P}(1+\varpi)+\Psi_6\mathcal{R}(1+\varpi)}{(\mathcal{R}-\mathcal{P})(1+\varpi)}\\ &= \frac{(1-\Psi_1)\mathcal{P}+\Psi_6\mathcal{R}}{(\mathcal{R}-\mathcal{P})} \qquad (59)\end{aligned}$$

From (53) to (56)

$$\Psi_2 = \frac{(\Psi_1-1)[\mathcal{P}(1+\varpi)+(\mathcal{T}+\mathcal{K})(3\varpi+2)]-\Psi_6[\mathcal{P}(1+\varpi)+(\mathcal{T}+\mathcal{K})(3\varpi+2)-\mathcal{R}(1+\varpi)]}{(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (60)$$

$$\Psi_3 = \frac{(\Psi_1-1)[\mathcal{L}+\mathcal{S}\varpi-\mathcal{P}(1+\varpi)]-\Psi_6[\mathcal{L}+\mathcal{S}\varpi-\mathcal{R}(1+\varpi)]+(\mathcal{R}-\mathcal{P})(1+\varpi)}{(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (61)$$

$$\Psi_4 = \frac{(\Psi_1-1)[2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi-2\mathcal{P}(1+\varpi)]-\Psi_6[2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi-2\mathcal{R}(1+\varpi)]}{(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (62)$$

$$\Psi_5 = \frac{(\Psi_1-1)[\mathcal{L}+\mathcal{S}(1+\varpi)+\mathcal{T}\varpi-2\mathcal{P}(1+\varpi)]-\Psi_6[\mathcal{L}+\mathcal{S}(1+\varpi)+\mathcal{T}\varpi-2\mathcal{R}(1+\varpi)]}{(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (63)$$

Therefore, Player II 's payoff will be

$$\begin{aligned}Pay_2 + Pay_3 &= -\frac{\kappa}{\eta}\\ &= -2\frac{[(1-\Psi_1)\mathcal{P}+\Psi_6\mathcal{R}](1+\varpi)}{(\Psi_1-1)-\Psi_6}\\ &= 2\frac{[(1-\Psi_1)\mathcal{P}+\Psi_6\mathcal{R}](1+\varpi)}{-\Psi_6-(\Psi_1-1)} \qquad (64)\end{aligned}$$

Similar for get Pay_1, we put $\eta = \xi = 0$

Then,

$$\Psi_1 - 1 = \lambda(\mathcal{R}(1+\varpi))+\kappa \qquad (66)$$

$$\Psi_2 - 1 = \lambda(\frac{2\mathcal{K}(1+\varpi)+\mathcal{T}\varpi}{2})+\kappa \qquad (67)$$

$$\Psi_3 - 1 = \lambda(\mathcal{S}+\mathcal{T}\varpi)+\kappa \qquad (68)$$

$$\Psi_4 = \lambda(\frac{2\mathcal{T}(1+\varpi)+\mathcal{K}\varpi}{2})+\kappa \qquad (69)$$

$$\Psi_5 = \lambda(\mathcal{L}+\mathcal{S}\varpi)+\kappa \qquad (70)$$

$$\Psi_6 = \lambda(\mathcal{P}(1+\varpi))+\kappa \qquad (71)$$

So,

$$\lambda = \frac{(\Psi_1-1)-\Psi_6}{(\mathcal{R}-\mathcal{P})(1+\varpi)} \qquad (72)$$

$$\begin{aligned}\kappa &= \frac{(1-\Psi_1)\mathcal{P}(1+\varpi)+\Psi_4\mathcal{R}(1+\varpi)}{(\mathcal{R}-\mathcal{P})(1+\varpi)}\\ &= \frac{(1-\Psi_1)\mathcal{P}+\Psi_6 R}{(\mathcal{R}-\mathcal{P})} \qquad (73)\end{aligned}$$

From (67) to (70)

$$\Psi_2 = \frac{(\Psi_1 - 1)[2(\mathcal{K} - \mathcal{P})(1+\varpi) + \mathcal{T}\varpi] - \Psi_6[2(\mathcal{K} - \mathcal{R})(1+\varpi) + \mathcal{T}\varpi] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{2(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (74)$$

$$\Psi_3 = \frac{(p_1 - 1)[\mathcal{S} + \mathcal{T}\varpi - \mathcal{P}(1+\varpi)] - p_6[\mathcal{S} + \mathcal{T}\varpi - \mathcal{R}(1+\varpi)] + (\mathcal{R} - \mathcal{P})(1+\varpi)}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (75)$$

$$\Psi_4 = \frac{(\Psi_1 - 1)[2(\mathcal{T} - \mathcal{P})(1+\varpi) + \mathcal{K}\varpi] - \Psi_6[2(\mathcal{T} - \mathcal{R})(1+\varpi) + \mathcal{K}\varpi]}{2(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (76)$$

$$\Psi_5 = \frac{(\Psi_1 - 1)[\mathcal{L} + \mathcal{S}\varpi - \mathcal{P}(1+\varpi)] - \Psi_6[\mathcal{L} + \mathcal{S}\varpi - \mathcal{R}(1+\varpi)]}{(\mathcal{R} - \mathcal{P})(1+\varpi)} \quad (77)$$

So, player I 's payoff will be

$$\begin{aligned} Pay_1 &= -\frac{\kappa}{\lambda} \\ &= -\frac{(1-\Psi_1)\mathcal{P}(1+\varpi) + \Psi_6\mathcal{R}(1+\varpi)}{(\Psi_1 - 1) - \Psi_6} \\ &= -\frac{[(1-\Psi_1)\mathcal{P} + \Psi_6\mathcal{R}](1+\varpi)}{(\Psi_1 - 1) - \Psi_6} \\ &= \frac{[(1-\Psi_1)\mathcal{P} + \Psi_6\mathcal{R}](1+\varpi)}{\Psi_6 - (\Psi_1 - 1)} \end{aligned} \quad (78)$$

By the same manner in section 2 for two players, we can conclude the conditions for Ψ_1 and Ψ_6 for three players game. Moreover, we can perform competition between players and infer the payoff in the presence of a relationship between players.

4. Conclusion

In the presence of a relationship between players, we discussed recurrent tactics and calculated the equivalent payoff between them. Due to the Prisoner's Game's significance in game theory, we concentrated on researching it and spoke about two and three players Iterated Prisoner's Dilemma games. Direct reciprocity is also a technique for maintaining mutual cooperation in games of recurring social dilemmas, where a player would maintain cooperation to avoid facing future retaliation from a co-player. known as a zero-determinant. The findings are as follows: First, we present the forms of solutions that extend the known results for infinitely repeated games with a relatedness factor ϖ. Second, for the most prominent ZD strategies, the **Equalizer** strategies, we have analytically derived the limits of single-move equivalent ZD strategies.

Conflicts of interest

Regarding the publishing of this paper, the authors state that they have no conflicts of interest.

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