

The $\Delta I = 1/2$ Rule and $\widehat{B}_K$ at $O(p^4)$ in the Chiral Expansion.

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ABSTRACT: We calculate the hadronic matrix elements to $O(p^4)$ in the chiral expansion for the $(\Delta S = 1)$ $K^0 \rightarrow 2\pi$ decays and the $(\Delta S = 2)$ $\bar{K}^0$ - K^0 oscillation. This is done within the framework of the chiral quark model. The chiral coefficients thus determined depend on the values of the quark and gluon condensates and the constituent quark mass. We show that it is possible to fit the $\Delta I = 1/2$ rule of kaon decays with values of the condensates close to those obtained by QCD sum rules. The renormalization invariant amplitudes are obtained by matching the hadronic matrix elements and their chiral corrections to the short-distance NLO Wilson coefficients. For the same input values, we study the parameter $\widehat{B}_K$ of kaon oscillation and find $\widehat{B}_K = 1.1 \pm 0.2$. As an independent check, we determine $\widehat{B}_K$ from the experimental value of the K_L - K_S mass difference by using our calculation of the long-distance contributions. The destructive interplay between the short- and long-distance amplitudes yields $\widehat{B}_K = 1.2 \pm 0.1$, in agreement with the direct calculation.

KEYWORDS: Kaon Physics, Chiral Lagrangians, Phenomenological Models.

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1 Introduction

The physics of kaons is an important testing ground for our understanding of low-energy QCD. The hadronic matrix elements for the nonleptonic decays must satisfy the non-trivial constraint of the $\Delta I = 1/2$ rule that shows the striking enhancement of the decay width in which the two final pions combine in the isospin $I = 0$ state with respect to the $I = 2$ state. The interplay between the renormalization-group evolution of the Wilson coefficients and the matrix elements of the relevant quark operators in both $\bar{K}^0$ - K^0 mixing and $K^0 \rightarrow 2\pi$ decays determines the theoretical predictions of the CP -violating parameters ε and ε'/ε , the knowledge of which is essential to the standard model.

The recent progress in the next-to-leading-order (NLO) computation of the Wilson coefficients of the $\Delta S = 1$ [1] and $\Delta S = 2$ [2] effective lagrangian at the quark level makes more urgent the need to bring under better control the non-perturbative estimate of the corresponding hadronic matrix elements. Different approaches have been pursued, ranging from lattice simulations [3], $1/N_c$ -expansion [4], phenomenological approaches [5] dispersion-integral techniques [6] and low-energy QCD modeling [7, 8]. A particular case of the latter, the chiral quark model (χ QM) [9, 10] has been analyzed in detail in a series of papers [11, 12, 13] with the encouraging result of fitting the $\Delta I = 1/2$ rule by a consistent and reasonable choice of the input parameters. These are essentially three: the quark condensate, the gluon condensate and M —a parameter of the model which corresponds to a typical constituent quark mass.

In a previous paper [11], we have calculated the coefficients of all the $O(p^2)$ terms of the $\Delta S = 1$ chiral lagrangian and included the corresponding chiral loop renormalizations (which are of $O(p^4)$). The present paper completes our previous analyses

- by including in the hadronic matrix elements the complete NLO $O(p^4)$ corrections;
- by updating the short-distance calculation of the Wilson coefficients according to the most recent determinations of α_s and m_t .

The determination of the $K \rightarrow 2\pi$ matrix elements at $O(p^4)$ is a non-trivial computation. To our knowledge it is the first time that the whole of these hadronic matrix elements are estimated to this order by any technique at all (for previous discussion, see refs. [14, 15]).

The introduction of these corrections allows us to determine a consistent range of values of the input parameters for which the $\Delta I = 1/2$ rule is reproduced with an accuracy at the 20% level. We find the best fit to this order in the chiral expansion for values of the condensates close to those derived by QCD sum rules and a value of M in agreement with independent estimates based on radiative kaon decays [16]. A coherent picture is thus provided: all hadronic matrix elements are computable for a common set of input parameters and no ad-hoc assumption is necessary to fit the $\Delta I = 1/2$ rule.

The hadronic matrix elements thus found are eventually matched together with one-loop chiral corrections to the corresponding NLO Wilson coefficients. The matching scale is a delicate choice in so far as it can neither be too high, in order for chiral perturbation theory to remain valid, nor too low, for the Wilson coefficients to be reliable. We choose a scale of 0.8 GeV that we identify with the chiral symmetry breaking scale. Even though at this scale α_s is already rather large, we have verified that the computed observables change by no more than 30% when moving from LO to NLO order Wilson coefficients. By considering the shift in the value of α_s at $\mu = 0.8$ GeV when going from the NLO to the next order, we can estimate that the neglect of next-to-NLO corrections in the running of the Wilson coefficients may affect our predictions at the 10% level, which is within the “systematic” error we assign to our analysis.

Once the input parameters of the model have been fixed, other quantities of kaon physics can also be estimated. In particular we discuss the deviation from the vacuum

insertion approximation (VSA) in $\bar{K}^0$ - K^0 oscillations that is parameterized by the scale independent parameter $\hat{B}_K$. The inclusion of the $O(p^4)$ corrections to the $\Delta S = 2$ chiral lagrangian provides us—for the input parameters identified by the $\Delta I = 1/2$ selection rule—with a central value of $\hat{B}_K \simeq 1.1$.

Contrarily to what one might think, such a large value is not in conflict with the experimental determination of the K_L - K_S mass difference ΔM_{LS} . To prove this point, we have computed the long-distance mesonic contributions that arise from the double insertion of the $\Delta S = 1$ chiral lagrangian, and find that they are about 20–30% of the whole mass difference and of the opposite sign with respect to the short-distance part [17]. By requiring that the ΔM_{LS} thus calculated fits the experimental value, we obtain $\hat{B}_K \simeq 1.2$ in striking agreement with the direct calculation.

We thus present two independent estimates of $\hat{B}_K$ within the χ QM approach:

- $\hat{B}_K = 1.1 \pm 0.2$ from direct calculation,
- $\hat{B}_K = 1.2 \pm 0.1$ from ΔM_{LS} , including long-distance contributions,

where the errors include a flat variation of all relevant input parameters. According to these results, values of $\hat{B}_K$ smaller than one are disfavored in the χ QM.

These results can then be applied to the prediction of the direct CP violating parameter ε'/ε . Because of the importance and the complexity of such a calculation, we have decided to present it in an independent paper [18].

In order for the present paper to be as self-contained as possible, we have included in the following two subsections the relevant lagrangians and a brief introduction to the χ QM. Such introductions summarize those of our previous papers, to which we refer the reader for further details and references.

1.1 The Quark Effective Lagrangian

Let us introduce our notation by recalling that the $\Delta S = 1$ quark effective lagrangian at a scale $\mu < m_c$ can be written as [19]

$$\mathcal{L}_{\Delta S=1} = -C_i(\mu) Q_i(\mu) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i \left[z_i(\mu) + \tau y_i(\mu) \right] Q_i(\mu). \quad (1.1)$$

The Q_i are effective four-quark operators obtained by integrating out in the standard model the vector bosons and the heavy quarks t , b and c . A convenient and by now

standard basis includes the following twelve quark operators:

$$\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V-A} , \\
Q_2 &= (\bar{s}u)_{V-A} (\bar{u}d)_{V-A} , \\
Q_{3,5} &= (\bar{s}d)_{V-A} \sum_q (\bar{q}q)_{V\mp A} , \\
Q_{4,6} &= (\bar{s}_\alpha d_\beta)_{V-A} \sum_q (\bar{q}_\beta q_\alpha)_{V\mp A} , \\
Q_{7,9} &= \frac{3}{2} (\bar{s}d)_{V-A} \sum_q \hat{e}_q (\bar{q}q)_{V\pm A} , \\
Q_{8,10} &= \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_q \hat{e}_q (\bar{q}_\beta q_\alpha)_{V\pm A} , \\
Q_{11} &= \frac{g_s}{16\pi^2} \bar{s} [m_d (1 + \gamma_5) + m_s (1 - \gamma_5)] \sigma \cdot G d , \\
Q_{12} &= \frac{e}{16\pi^2} \bar{s} [m_d (1 + \gamma_5) + m_s (1 - \gamma_5)] \sigma \cdot F d ,
\end{aligned} \tag{1.2}$$

where α, β denote color indices ($\alpha, \beta = 1, \dots, N_c$) and $\hat{e}_q$ are quark charges. Color indices for the color singlet operators are omitted. The subscripts $(V \pm A)$ refer to $\gamma_\mu(1 \pm \gamma_5)$. We recall that $Q_{1,2}$ stand for the W -induced current–current operators, Q_{3-6} for the QCD penguin operators and Q_{7-10} for the electroweak penguin (and box) ones. The quark operators $Q_{11,12}$, involving the gluon and photon fields, are the dipole penguin operators whose $K^0 \rightarrow \pi\pi$ matrix elements arise at $O(p^4)$ [14].

Even though not all the operators in eq. (1.2) are independent, this basis is of particular interest for the present numerical analysis because it is that employed for the calculation of the Wilson coefficients to the NLO in α_s [1].

In the present paper we will mainly discuss the features related to the first six operators in (1.2) because the electroweak penguins Q_{7-10} have their contributions suppressed by the smallness of their CP conserving Wilson coefficients, while Q_{11-12} have very small matrix elements [14]. The electroweak operators play however a crucial role in the discussion of ε'/ε [18].

The functions $z_i(\mu)$ and $y_i(\mu)$ are the Wilson coefficients and V_{ij} the Kobayashi-Maskawa (KM) matrix elements; $\tau = -V_{td}V_{ts}^*/V_{ud}V_{us}^*$. The numerical values of the Wilson coefficients at a given scale depend on α_s . We take the most recent world average [20]

$$\alpha_s(m_Z) = 0.1189 \pm 0.0020 , \tag{1.3}$$

which at the NLO corresponds approximately to

$$\Lambda_{\text{QCD}}^{(4)} = 340 \pm 40 \text{ MeV} . \tag{1.4}$$

We match the Wilson coefficients at the m_W scale with the full electroweak theory by using the LO $\overline{MS}$ running top mass $m_t(m_W) = 177 \pm 7 \text{ GeV}$ which corresponds to the pole mass $m_t^{\text{pole}} = 175 \pm 6 \text{ GeV}$ [21]. For the remaining quark thresholds we take $m_b(m_b) = 4.4 \text{ GeV}$ and $m_c(m_c) = 1.4 \text{ GeV}$.

Similarly, the effective $\Delta S = 2$ quark lagrangian at scales $\mu < m_c$ can be written as

$$\begin{aligned}
\mathcal{L}_{\Delta S=2} &= -C_{S2}(\mu) Q_{S2}(\mu) \\
&= -\frac{G_F^2 m_W^2}{4\pi^2} \left[\lambda_c^2 \eta_1 S(x_c) + \lambda_t^2 \eta_2 S(x_t) + 2\lambda_c \lambda_t \eta_3 S(x_c, x_t) \right] b(\mu) Q_{S2}(\mu) \tag{1.5}
\end{aligned}$$

where $\lambda_j = V_{jd}V_{js}^*$ and $x_i = m_i^2/m_W^2$. We denote by Q_{S2} the $\Delta S = 2$ local four quark operator

$$Q_{S2} = (\bar{s}_L \gamma^\mu d_L)(\bar{s}_L \gamma_\mu d_L), \quad (1.6)$$

which is the only numerically relevant operator in this case.

The integration of the electroweak loops leads to the functions $S(x)$ and $S(x_c, x_t)$, the exact expressions of which can be found in ref. [22]. They describe the dependence of the $\Delta S = 2$ transition amplitude on the masses of the charm and top quarks in the absence of strong interactions.

The short-distance QCD corrections are encoded in the coefficients η_1 , η_2 and η_3 and $b(\mu)$. The η_i coefficients represent the renormalization effects down to the scale m_c . They are functions of the heavy quarks masses and of Λ_{QCD} . These QCD corrections are available to NLO in the strong coupling [2]. The scale-dependent function $b(\mu)$ describes the overall running below the charm threshold and it is given by

$$b(\mu) = [\alpha_s(\mu)]^{-2/9} \left(1 - J_3 \frac{\alpha_s(\mu)}{4\pi} \right), \quad (1.7)$$

where J_3 depends on the γ_5 -scheme used in the regularization. The naive dimensional regularization (NDR) and the 't Hooft-Veltman (HV) scheme yield, respectively:

$$J_3^{\text{NDR}} = -\frac{307}{162} \quad \text{and} \quad J_3^{\text{HV}} = -\frac{91}{162}. \quad (1.8)$$

1.2 The Chiral Quark Model

In order to evaluate the bosonization of the quark operators in eqs. (1.2) and (1.6) we exploit the χ QM approach which provides an effective link between QCD and chiral perturbation theory.

The χ QM can be thought of as the mean field approximation of the extended Nambu-Jona-Lasinio (ENJL) model for low-energy QCD. A detailed discussion of the ENJL model and its relationship with QCD—as well as with the χ QM—can be found, for instance, in ref. [8].

In the χ QM, the light (constituent) quarks are coupled to the Goldstone mesons by the term

$$\mathcal{L}_{\chi\text{QM}}^{\text{int}} = -M \left(\bar{q}_R \Sigma q_L + \bar{q}_L \Sigma^\dagger q_R \right), \quad (1.9)$$

where $q^T \equiv (u, d, s)$ is the quark flavor triplet, and the 3×3 matrix

$$\Sigma \equiv \exp \left(\frac{2i}{f} \Pi(x) \right) \quad (1.10)$$

contains the pseudoscalar octet $\Pi(x) = \sum_a \lambda^a \pi^a(x)/2$, ($a = 1, \dots, 8$). The scale f is identified at the tree level with the pion decay constant f_π (and equal to f_K before chiral loops and higher order corrections are introduced). In the χ QM, the pion decay constant is given by a logarithmic divergent quark loop integral $f_\pi^{(0)}$, which is numerically identified with f , such that $f_+ \equiv f_\pi^{(0)}/f = 1$.

The χ QM has been discussed in several works over the years [9, 10]. We opted for the somewhat more restrictive definition suggested in ref. [10] (and there referred to as the QCD effective action model) in which the meson degrees of freedom do not propagate in the original lagrangian.

The QCD gluonic fields are considered as integrated out down to the chiral breaking scale Λ_χ , here acting as an infrared cut-off. The effect of the remaining low-frequency modes are assumed to be well-represented by gluonic vacuum condensates, the leading contribution coming from

$$\left\langle \frac{\alpha_s}{\pi} GG \right\rangle. \quad (1.11)$$

The constituent quarks are taken to be propagating in the fixed background of the soft gluons. This defines an effective QCD lagrangian $\mathcal{L}_{\text{QCD}}^{\text{eff}}(\Lambda_\chi)$, whose propagating fields are the u, d, s quarks. The strong χ QM lagrangian is therefore given by

$$\mathcal{L}_{\chi\text{QM}} = \mathcal{L}_{\text{QCD}}^{\text{eff}}(\Lambda_\chi) + \mathcal{L}_{\chi\text{QM}}^{\text{int}}. \quad (1.12)$$

The $\mathcal{L}_{\chi\text{QM}}$ interpolates between the chiral breaking scale Λ_χ and M (the constituent quark mass). The three light quarks (u, d, s) are the only dynamical degrees of freedom present within this range. The total quark masses are given by $M + m_q$ where m_q is the current quark mass in the QCD lagrangian. The Goldstone bosons and the soft QCD gluons are taken in our approach as external fields. A kinetic term for the mesons, as well as the complete chiral lagrangian, is generated and determined by integrating out the constituent quark degrees of freedom of the model. By combining eq. (1.12) with eqs. (1.1) and (1.6) one may obtain the $\Delta S = 1$ and $\Delta S = 2$ weak chiral lagrangians as effective theories of the χ QM. In the matching process, the many coefficients of the chiral lagrangian are determined—to the order $O(\alpha_s N_c)$ in our computation—in terms of M , the quark and gluon condensates. We neglect heavier scalar, vector and axial meson multiplets.

In conventional chiral perturbation theory the scale dependence of meson loops renormalization is canceled by construction by the $O(p^4)$ counterterms in the chiral lagrangian. While in our approach this is maintained for the strong sector of the chiral lagrangian, the tree-level counterterms of the weak sector are taken to be μ independent and a scale dependence is introduced in the hadronic matrix elements via the meson loops, evaluated in dimensional regularization with minimal subtraction. This scale dependence is eventually matched with that of the Wilson coefficients. Tree level counterterms acquire a scale dependence at the next order in the chiral expansion, via meson loop renormalization.

A more detailed discussion about our approach including the complete determination of the $\Delta S = 1$ chiral lagrangian at $O(p^2)$ can be found in refs. [11, 14].

2 Building Blocks

Within the χ QM, matrix elements of the quark operators Q_{1-10} can be calculated in the factorizable approximation as products of two currents or two densities. Non-factorizable

matrix-elements are proportional to the gluon condensate. In this way the hadronic matrix elements of the relevant operators are constructed in terms of products of two building blocks. These elementary blocks must be computed to the appropriated order in p^2 and m_q to yield the $O(p^4)$ matrix elements.

By integrating over quark loops we obtain the following matrix elements of quark densities, based on the lagrangian in eq. (1.12):

$$\begin{aligned} \langle 0 | \bar{s} \gamma_5 u | K^+(k) \rangle &= i\sqrt{2} \left\{ \frac{\langle \bar{q}q \rangle}{f} - k^2 \frac{f}{2M} \left(f_+ + \frac{k^2}{\Lambda_\chi^2} \right) \right. \\ &\quad + (m_s + m_u) f \left(f_+ + 3 \frac{k^2}{\Lambda_\chi^2} \right) \\ &\quad \left. + \frac{f}{M} \left[(m_s^2 + m_u^2) \left(f_+ - 6 \frac{M^2}{\Lambda_\chi^2} \right) - m_s m_u f_+ \right] \right\}, \quad (2.1) \end{aligned}$$

$$\begin{aligned} \langle \pi^+(p_+) | \bar{s} d | K^+(k) \rangle &= -\frac{\langle \bar{q}q \rangle}{f^2} + \frac{q^2}{2M} f_+ + \frac{3M}{2\Lambda_\chi^2} (P^2 - q^2) \\ &\quad + \frac{1}{16M\Lambda_\chi^2} (P^4 + 2P^2 q^2 + 5q^4) \\ &\quad - (m_s + m_d) \left[f_+ + 2 \frac{q^2}{\Lambda_\chi^2} \right] - (m_s - m_d) \frac{q \cdot P}{\Lambda_\chi^2} \\ &\quad - 2m_u \left[f_+ + \frac{3P^2 + 5q^2}{4\Lambda_\chi^2} \right] \\ &\quad - \frac{1}{M} \left[(m_s^2 + m_s m_d + m_d^2) \left(f_+ - 6 \frac{M^2}{\Lambda_\chi^2} \right) \right. \\ &\quad \left. - 6m_u (m_s + m_d + m_u) \frac{M^2}{\Lambda_\chi^2} \right], \quad (2.2) \end{aligned}$$

where $\langle \bar{q}q \rangle$ is the quark condensate for zero current quark mass. In the calculation of the four-fermion matrix elements at $O(p^4)$ also the density amplitude $\langle \pi\pi | \bar{s} \gamma_5 d | K^0 \rangle$ contributes. A discussion on its evaluation is given in appendix A.

The corresponding quark current matrix elements are given by:

$$\begin{aligned} \langle 0 | \bar{s} \gamma^\mu (1 - \gamma_5) u | K^+(k) \rangle &= -i\sqrt{2} f \left\{ f_+ + \frac{k^2}{\Lambda_\chi^2} \right. \\ &\quad \left. + \frac{m_s + m_u}{2M} \left[f_+ - 12 \frac{M^2}{\Lambda_\chi^2} \right] \right\} k^\mu \quad (2.3) \end{aligned}$$

$$\begin{aligned} \langle \pi^+(p_+) | \bar{s} \gamma^\mu (1 - \gamma_5) d | K^+(k) \rangle &= - \left[f_+ + \frac{P^2 + 3q^2}{2\Lambda_\chi^2} \right] P^\mu + \frac{q \cdot P}{\Lambda_\chi^2} q^\mu \\ &\quad + \frac{3M(m_s + m_d + 2m_u)}{\Lambda_\chi^2} P_\mu \\ &\quad - \frac{m_s - m_d}{2M} \left[f_+ - 6 \frac{M^2}{\Lambda_\chi^2} \right] q^\mu, \quad (2.4) \end{aligned}$$

$$\langle 0 | \bar{s} \gamma^\mu T^a (1 - \gamma_5) u | K^+(k) \rangle = -\frac{ig_s \sqrt{2}}{16\pi^2 f} G_{\nu\tau}^a A_G^{\mu\nu\tau}(k) \quad (2.5)$$

$$\langle \pi^+(p_+) | \bar{s} \gamma^\mu T^a (1 - \gamma_5) d | K^+(k) \rangle = -\frac{g_s}{16\pi^2 f^2} G_{\nu\tau}^a B_G^{\mu\nu\tau}(q, P) \quad (2.6)$$

where $q = k - p_+$ and $P = k + p_+$, k being the incoming momentum of the kaon field and p_+ the outgoing momentum of the pion field. The chiral symmetry breaking scale is taken as $\Lambda_\chi = 2\pi\sqrt{6/N_c}f \simeq 0.8$ GeV, while $f_+ \equiv f_\pi^{(0)}/f = 1$ is the vector form factor at zero momentum transfer. Finally, $G_{\nu\tau}^a$ with $a = 1, \dots, 8$ is the usual $SU(3)_c$ gluon field tensor and T^a are the $SU(3)_c$ generators normalized as $\text{Tr } T^a T^b = \delta^{ab}/2$.

The eqs. (2.5)–(2.6) represent the gluonic corrections, which are computed by using one-gluon dressed quark propagators and color Fierz transformations on the four-quark operators (the $SU(3)_c$ index a is to be summed over in the full matrix element). They contribute to the non-factorizable part of the hadronic matrix elements via the relation

$$g_s^2 G_{\mu\nu}^a G_{\alpha\beta}^a = \frac{\pi^2}{3} \langle \frac{\alpha_s}{\pi} GG \rangle (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}) \quad (2.7)$$

which allows us to express our results in terms of the gluonic vacuum condensate.

The two gluonic form factors are given by:

$$\begin{aligned} A_G^{\mu\nu\tau}(k) &= \epsilon(k, \mu, \nu, \tau) \left[1 + \frac{k^2}{6M^2} - \frac{m_s + m_u}{2M} \right] \\ &+ i \frac{m_s - m_u}{4M} [k_\nu g_{\mu\tau} - k_\tau g_{\mu\nu}] \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} B_G^{\mu\nu\tau}(q, P) &= \epsilon(P, \mu, \nu, \tau) \left[1 + \frac{P^2}{12M^2} + \frac{q^2}{4M^2} \right] \\ &- \epsilon(q, \mu, \nu, \tau) \frac{q \cdot P}{6M^2} - \epsilon(P, q, \mu, \nu) \frac{q_\tau}{12M^2} + \epsilon(P, q, \mu, \tau) \frac{q_\nu}{12M^2} \\ &- i \frac{2}{3} (q_\nu g_{\mu\tau} - q_\tau g_{\mu\nu}) - i \frac{P_\mu}{30M^2} (P_\nu q_\tau - P_\tau q_\nu) \\ &+ i \frac{7q \cdot P}{120M^2} (P_\nu g_{\mu\tau} - P_\tau g_{\mu\nu}) - i \left[\frac{29P^2}{240M^2} + \frac{11q^2}{80M^2} \right] (q_\nu g_{\mu\tau} - q_\tau g_{\mu\nu}) \\ &- \frac{m_s}{12M} \left\{ \epsilon[(5P + q), \mu, \nu, \tau] + \frac{i}{4} [(P_\nu - 19q_\nu) g_{\mu\tau} - (\nu \rightarrow \tau)] \right\} \\ &- \frac{m_d}{12M} \left\{ \epsilon[(5P - q), \mu, \nu, \tau] - \frac{i}{4} [(P_\nu + 19q_\nu) g_{\mu\tau} - (\nu \rightarrow \tau)] \right\} \\ &- \frac{m_u}{12M} \left\{ 12\epsilon(P, \mu, \nu, \tau) - \frac{i}{4} [14q_\nu g_{\mu\tau} - (\nu \rightarrow \tau)] \right\} \end{aligned} \quad (2.9)$$

where $\epsilon(k, \mu, \nu, \tau) \equiv k^\alpha \epsilon_{\alpha\mu\nu\tau}$.

The procedure by which it is possible to determine the hadronic matrix elements and the corresponding chiral coefficient to the desired order for all the relevant operators by means of the given building blocks is discussed in refs. [11, 14], to which we refer the interested reader.

Before proceeding it is important to identify what parts of the building blocks are absorbed by the renormalization of the parameters and fields of the lagrangian. To this we now turn.

3 Wave-Function and Coupling-Constant Renormalizations

To the order $O(p^4)$ at which we are working, both the wave-function and the meson decay constant renormalizations must be included. They give sizeable contributions that cannot be neglected. In our computation, the renormalization comes in two parts which are conceptually distinct. On the one hand we have the chiral renormalization generated by the meson loops, to which we have to add the renormalization that is specific of the χ QM and originates from the expansion of the building blocks beyond the leading order. In refs. [11, 12, 13] only the chiral loop renormalizations were included.

3.1 Wave-Function Renormalizations

The wave-function renormalizations which arise in the χ QM from direct calculation of the $K \rightarrow K$ and $\pi \rightarrow \pi$ propagators are given at $O(p^2)$ by:

$$\begin{aligned} Z_K &= 1 - 2\frac{m_K^2}{\Lambda_\chi^2} + 6\frac{M(m_s + m_u)}{\Lambda_\chi^2} \\ Z_\pi &= 1 - 2\frac{m_\pi^2}{\Lambda_\chi^2} + 6\frac{M(m_d + m_u)}{\Lambda_\chi^2} \end{aligned} \quad (3.1)$$

The complete $O(p^4)$ expressions are given in appendix B. The renormalizations above are added to those induced by the one-loop chiral corrections.

3.2 Renormalization of f_K and f_π in the χ QM

The building block in eq. (2.3) gives directly the corrected (unrenormalized) version of the coupling constant f yielding (for instance for f_K)

$$f_K^U = f \left[1 + \frac{m_K^2}{\Lambda_\chi^2} + \frac{m_s + m_u}{2M} \left(1 - 12\frac{M^2}{\Lambda_\chi^2} \right) \right] \quad (3.2)$$

However, it is only after inclusion of the wave-function renormalization in eq. (3.1) that we find the complete χ QM expression:

$$f_K^R = \sqrt{Z_K} f_K^U = f \left[1 + \frac{m_s + m_u}{2M} \left(1 - 6\frac{M^2}{\Lambda_\chi^2} \right) \right] \quad (3.3)$$

and similarly for f_π . Notice that $f_{\pi,K}^R$ coincide at $O(p^4)$ with the physical decay constants $f_{\pi,K}$ only after inclusion of the proper chiral loop renormalizations (see next subsection).

The consistency of the entire procedure can be verified by considering the renormalization of the charged form factor $f_+(q^2)$ in the coupling of mesons to an external vector

field (photon or vector boson). This form factor can be read off eq. (2.4). After renormalization we find that the on-shell form factor is

$$f_+^R(q^2) = \sqrt{Z_K} \sqrt{Z_\pi} f_+^U(q^2) = 1 + \frac{q^2}{\Lambda_\chi^2}, \quad (3.4)$$

which correctly preserves the current-conservation condition $f_+(0) = 1$.

By matching the expressions above for the renormalized f_K and f_π with the chiral lagrangian results we obtain the χ QM determinations (see also appendix B)

$$L_4 = 0, \quad (3.5)$$

$$L_5 = -\frac{f^4}{8M\langle\bar{q}q\rangle} \left(1 - 6\frac{M^2}{\Lambda_\chi^2}\right), \quad (3.6)$$

where by PCAC

$$\langle\bar{q}q\rangle = -\frac{f^2 m_K^2}{m_s + m_u} = -\frac{f^2 m_\pi^2}{m_d + m_u}. \quad (3.7)$$

The above equations are obtained at the constituent quark mass scale M , where the quark fields are integrated out and the standard chiral lagrangian arise as the effective theory of the χ QM. The parameter L_4 is vanishing in the χ QM up to higher order gluon condensate contributions. The usual numerical determinations of the renormalized L_4 and L_5 in chiral perturbation theory must then be run down to this scale in order to be compared to the result of the χ QM, after taking into account the different subtraction used in the renormalization of the chiral loops. We address this issue in the next subsection.

3.3 The Chiral Coupling f at the One-Loop Order

The one-loop expressions for f_π and f_K allow us to determine the value of the chiral coupling f to be used in the NLO calculation. In order to compare eq. (3.3), and the analogous one for f_π , with the physical values of the decay constants, the $O(p^4)$ renormalization induced by chiral loops has to be included. In terms of the tree-level counterterms of the $O(p^4)$ strong chiral lagrangian, one obtains the equations [23]

$$f^2 - f_\pi f - 2h(m_\pi, \mu) - h(m_K, \mu) + 4m_\pi^2 L_5(\mu) + (4m_\pi^2 + 8m_K^2) L_4(\mu) = 0, \quad (3.8)$$

for the pion decay constant f_π , and

$$f^2 - f_K f - \frac{3}{4}h(m_\pi, \mu) - \frac{3}{2}h(m_K, \mu) - \frac{3}{4}h(m_\eta, \mu) + 4m_K^2 L_5(\mu) + (4m_\pi^2 + 8m_K^2) L_4(\mu) = 0, \quad (3.9)$$

for f_K , where, by keeping only the logarithmic part from the loop integrals,

$$h(m, \mu) = \frac{m^2}{32\pi^2} \ln \frac{m^2}{\mu^2}. \quad (3.10)$$

This prescription defines the renormalized couplings L_5 and L_4 . Once L_4 is given, these equations may be solved for L_5 and f .

The entire procedure contains many uncertainties, the input value of L_4 being one of them. According to the original derivation of ref. [23] we take $L_4(m_\eta) = 0.0 \pm 0.5$. Using the known anomalous dimensions of the couplings L_i [23], we run L_4 up to the matching scale $\mu = \Lambda_\chi \simeq 0.8$ GeV where we find

$$L_4(\Lambda_\chi) = -(0.3 \pm 0.5) \times 10^{-3}. \quad (3.11)$$

For this value, the solution of the system of eq. (3.8) and eq. (3.9) at the scale $\mu = \Lambda_\chi$ gives

$$L_5(\Lambda_\chi) = (1.2 \pm 0.4) \times 10^{-3} \quad \text{and} \quad f = 0.086 \pm 0.013 \text{ GeV}, \quad (3.12)$$

where the errors come from uncertainties in the input parameters, mainly those for L_4 .

By solving the same equations at the scale $M \simeq 0.2$ GeV, where

$$L_4(M) = (0.8 \pm 0.5) \times 10^{-3}, \quad (3.13)$$

we obtain

$$L_5(M) = (4.3 \pm 0.4) \times 10^{-3} \quad (3.14)$$

and $f = 0.086 \pm 0.013$ GeV, as expected from the scale-independence of f .

A consistent solution of eqs. (3.8)–(3.9) is also obtained by using the $\overline{MS}$ result for $h(m, \mu)$, that is

$$h(m, \mu) = \frac{m^2}{32\pi^2} \left(\ln \frac{m^2}{\mu^2} - 1 \right). \quad (3.15)$$

In this respect, it is important to remark that the determination of L_5 and L_4 depend on the subtraction scheme employed in the renormalization of the chiral loops (while the value of f is by construction independent on it). Only by consistently adopting the same renormalization prescription the physical quantities remain invariant. By taking $f = 0.086 \pm 0.013$ GeV we obtain the following $\overline{MS}$ results:

$$\bar{L}_4(M) = (0.4^{+0.4}_{-0.6}) \times 10^{-3}, \quad (3.16)$$

$$\bar{L}_5(M) = (3.2 \pm 0.3) \times 10^{-3}, \quad (3.17)$$

and

$$\bar{L}_4(\Lambda_\chi) = (-0.7^{+0.4}_{-0.6}) \times 10^{-3}, \quad (3.18)$$

$$\bar{L}_5(\Lambda_\chi) = (0.1 \pm 0.3) \times 10^{-3}. \quad (3.19)$$

The value of L_4 in eq. (3.16) is consistent with the χ QM determination in eq. (3.5). The matching of the expression in eq. (3.6) with the result of eq. (3.17) gives a value of $\langle \bar{q}q \rangle$ at $\mu = M$ of $(-185 \pm 10 \text{ MeV})^3$. The larger values of the quark condensate that can be extracted at the scale Λ_χ from eq. (3.19) are consistent with the range of values of $\langle \bar{q}q \rangle$ that we obtain at $\mu = \Lambda_\chi$ from the fit of the $\Delta I = 1/2$ rule, computed in the $\overline{MS}$ scheme.

Even though the actual values for the parameters $\langle \bar{q}q \rangle$ and $\langle \alpha_s GG/\pi \rangle$ that fit the selection rule will vary depending on the choice of f in the range of eq. (3.12), we have

verified that the predictions of the other observables remain rather stable when varying the input parameters accordingly. In particular, the value of $\hat{B}_K$ varies by a few percents by changing f within the range of eq. (3.12), provided we use the corresponding values of the input parameters that fit the $\Delta I = 1/2$ rule. We will therefore take the central value of f as a fixed input in all numerical estimates.

4 Hadronic Matrix Elements

The chiral lagrangian coefficients to order $O(p^2)$ were computed in previous papers: refs. [11] and [17] for $\Delta S = 1$ and $\Delta S = 2$ transitions respectively. Here we report the NLO corrections to the matrix elements $\langle Q_i \rangle_I^{NLO}$ which are obtained by means of the building blocks in section 2. We define $m_u = m_d \equiv \widehat{m}$, the average of the u and d current quark masses, consistently to $m_{\pi^\pm} = m_{\pi^0} = m_\pi$.

There is no mass renormalization to this order because all masses in the matrix elements of the operators Q_{1-6} originate from the external momenta which are defined to be on the physical mass-shell. As expected, constant terms in the densities, which enter in the determination of the operators $Q_{5,6}$, cancel in the complete matrix elements.

The total matrix elements have the form

$$\langle Q_i(\mu) \rangle_I = Z_\pi \sqrt{Z_K} \left[\langle Q_i \rangle_I^{LO} + \langle Q_i(\mu) \rangle_I^{NLO} \right] + a_i^I(\mu), \quad (4.1)$$

where Q_i are the operators in eq. (1.2),

$$\langle Q_i \rangle_I \equiv \langle (\pi\pi)_I | Q_i | K^0 \rangle. \quad (4.2)$$

After including the wavefunction renormalization, the matrix elements are expanded to $O(p^4)$, discarding higher order terms. The functions $a_i^I(\mu)$ represent the scale dependent meson-loop corrections, including the mesonic wavefunction renormalization. They are defined as the isospin projections of the $a_i^{+-}(\mu)$ and $a_i^{00}(\mu)$ corrections computed in ref. [11], properly rescaled by factors of f_π/f in order to replace $f_\pi \rightarrow f$ in the NLO evaluation. The scale dependence of the NLO part of the matrix elements is a consequence of the perturbative scale dependence of the current quark masses which enter at this order.

The LO matrix elements $\langle Q_i \rangle_I^{LO}$ can be found in ref. [11], all occurrences of f_π being replaced by f . The NLO $I = 0$ and 2 contributions to the $K^0 \rightarrow \pi\pi$ matrix elements are given by:

$$\langle Q_1 \rangle_0^{NLO} = \frac{1}{3} X \left[\left(-1 + \frac{2}{N_c} \right) \beta - \frac{2}{N_c} \delta_{\langle GG \rangle} \beta_G \right], \quad (4.3)$$

$$\langle Q_1 \rangle_2^{NLO} = \frac{\sqrt{2}}{3} X \left[\left(1 + \frac{1}{N_c} \right) \beta - \frac{\delta_{\langle GG \rangle}}{N_c} \beta_G \right], \quad (4.4)$$

$$\langle Q_2 \rangle_0^{NLO} = -\frac{1}{3} X \left[\left(-2 + \frac{1}{N_c} \right) \beta - \frac{\delta_{\langle GG \rangle}}{N_c} (\beta_G + 3\gamma_G) \right], \quad (4.5)$$

$$\langle Q_2 \rangle_2^{NLO} = \langle Q_1 \rangle_2^{NLO}, \quad (4.6)$$

$$\langle Q_3 \rangle_0^{NLO} = \frac{1}{N_c} X \left[\beta - \delta_{\langle GG \rangle} (\beta_G - \gamma_G) \right], \quad (4.7)$$

$$\langle Q_4 \rangle_0^{NLO} = \langle Q_2 \rangle_0^{NLO} - \langle Q_1 \rangle_0^{NLO} + \langle Q_3 \rangle_0^{NLO}, \quad (4.8)$$

$$\langle Q_5 \rangle_0^{NLO} = \frac{2}{N_c} X \gamma \quad (4.9)$$

$$\langle Q_6 \rangle_0^{NLO} = 2X \left(\gamma + \frac{\delta_{\langle GG \rangle}}{N_c} \gamma_G \right) \quad (4.10)$$

where

$$X \equiv \sqrt{3} f (m_K^2 - m_\pi^2) \quad (4.11)$$

and the gluon-condensate correction $\delta_{\langle GG \rangle}$ is given by

$$\delta_{\langle GG \rangle} = \frac{N_c}{2} \frac{\langle \alpha_s GG / \pi \rangle}{16\pi^2 f^4}. \quad (4.12)$$

The quantities β, β_G, γ and γ_G are dimensionless functions of the mass parameters:

$$\begin{aligned} \beta &= \frac{m_K^2 + 2m_\pi^2}{\Lambda_\chi^2} - 3 \frac{M}{\Lambda_\chi^2} (m_s + 3\widehat{m}) \\ &\quad + \frac{m_s - \widehat{m}}{M} \left(1 - 6 \frac{M^2}{\Lambda_\chi^2} \right) \frac{m_\pi^2}{2(m_K^2 - m_\pi^2)} + \frac{\widehat{m}}{M} \left(1 - 12 \frac{M^2}{\Lambda_\chi^2} \right), \end{aligned} \quad (4.13)$$

$$\beta_G = \frac{m_K^2 + 2m_\pi^2}{6M^2} - \frac{5m_s + 17\widehat{m}}{12M} - \frac{(m_s - \widehat{m}) m_\pi^2}{12M (m_K^2 - m_\pi^2)} - \frac{\widehat{m}}{M}, \quad (4.14)$$

$$\begin{aligned} \gamma &= \frac{\langle \bar{q}q \rangle}{f^2} \left[\frac{m_K^2}{2M\Lambda_\chi^2} - \frac{m_K^2(2m_s + 6\widehat{m}) - m_\pi^2(m_s + 7\widehat{m})}{(m_K^2 - m_\pi^2)\Lambda_\chi^2} \right. \\ &\quad \left. + \frac{2\widehat{m}(m_s - \widehat{m})}{M(m_K^2 - m_\pi^2)} \left(2f_+ - 6 \frac{M^2}{\Lambda_\chi^2} \right) \right] \\ &\quad + f_+^2 \left[-\frac{m_K^2 + m_\pi^2}{4M^2} + \frac{(m_s + 5\widehat{m})}{2M} + \frac{2\widehat{m}(m_s - \widehat{m})}{m_K^2 - m_\pi^2} \right] \\ &\quad + \frac{3Mf_+}{\Lambda_\chi^2} \left[\frac{m_K^2}{2M} - \frac{m_K^2(m_s + 3\widehat{m}) - 2m_\pi^2(m_s + \widehat{m})}{(m_K^2 - m_\pi^2)} \right], \end{aligned} \quad (4.15)$$

$$\gamma_G = \frac{m_K^2}{m_K^2 - m_\pi^2} \frac{m_s - \widehat{m}}{6M}. \quad (4.16)$$

Similarly, for the $\Delta S = 2$ matrix element $\langle \bar{K}^0 | Q_{2S} | K^0 \rangle$ we write

$$\langle Q_{2S}(\mu) \rangle = Z_K \left[\langle Q_{2S} \rangle^{LO} + \langle Q_{2S}(\mu) \rangle^{NLO} \right] + a_{2S}(\mu), \quad (4.17)$$

with

$$\langle Q_{2S} \rangle^{LO} = m_K^2 f^2 \left[1 + \frac{1}{N_c} (1 - \delta_{\langle GG \rangle}) \right], \quad (4.18)$$

and

$$\begin{aligned} \langle Q_{2S}(\mu) \rangle^{NLO} &= m_K^2 f^2 \left\{ \left(1 + \frac{1}{N_c} \right) \left[2 \frac{m_K^2}{\Lambda_\chi^2} + \frac{m_s + \widehat{m}}{M} \left(1 - 12 \frac{M^2}{\Lambda_\chi^2} \right) \right] \right. \\ &\quad \left. - \frac{\delta_{\langle GG \rangle}}{N_c} \left(\frac{m_K^2}{3M^2} - \frac{m_s + \widehat{m}}{M} \right) \right\}, \end{aligned} \quad (4.19)$$

where the scale dependence enters through the perturbative running of the current quark masses.

The chiral-loop correction to $\langle \bar{K}^0 | Q_{2S} | K^0 \rangle$, including the meson wave-function renormalization, is given by

$$a_{2S}(\mu) = m_K^2 f_K^2 \left[1 + \frac{1}{N_c} (1 - \delta_{\langle GG \rangle}) \right] r_{2S}(\mu) , \quad (4.20)$$

where, having normalized the $O(p^4)$ amplitude to $m_K^2 f_K^2$,

$$r_{2S}(\mu) = 1.45 + 0.676 \ln \mu^2 , \quad (4.21)$$

and μ is in units of GeV. The given numbers depend only on the octet meson masses and are obtained for the values given in appendix C.

By subtracting the $\overline{MS}$ chiral loop renormalization of $f^2 \rightarrow f_K^2$ at the scale μ one obtains [17]

$$\bar{r}_{2S}(\mu) = 0.728 + 0.373 \ln \mu^2 , \quad (4.22)$$

which shows that a large part of the meson-loop correction goes into the renormalization of f_K .

Everywhere, in the NLO as well as in the leading order matrix elements, the coupling constant f must be understood as given by the values discussed in section 3.3. This replacement in the NLO terms is optional, given that the difference is of $O(p^6)$. For convenience, in our numerical analysis we replace all occurrences of f with its one-loop value.

Another remark concerns the quark condensate which appears in the building blocks of the quark densities. While all factorizable gluonic corrections are absorbed in the physical definition of the chiral coupling and the quark condensate, there are mass dependent contributions to the latter that we have not included in eq. (4.15). At the needed $O(m_q)$, the relation between the condensate $\langle \bar{q}q \rangle$ that we use as a parameter in our analysis, and the condensate $\langle \bar{q}q \rangle_{m_q}$ calculated at non-zero current quark masses is given by

$$\langle \bar{q}q \rangle_{m_q} = \langle \bar{q}q \rangle + \frac{m_q}{M} \left[\langle \bar{q}q \rangle + 2M f^2 f_+ \right] . \quad (4.23)$$

The correction is negligible for $q = u, d$ and remains small also for $\langle \bar{s}s \rangle$ due to a partial cancellation between the two terms in square brackets. In what follows we will always refer to the flavor independent parameter $\langle \bar{q}q \rangle$.

4.1 Chiral Loop Corrections

The one-loop chiral corrections $a_\ell(\mu)$ are included at this stage operator by operator on top of the χ QM estimate of the matrix elements, as discussed in refs. [11, 17]. Let us recall that throughout our analysis we use dimensional regularization and the modified minimal subtraction scheme as it is done for the Wilson coefficients. For this reason the numerical values for the $\Delta S = 1$ chiral loop corrections quoted in ref. [11] differ from those of ref. [15] where a non-minimal subtraction scheme is employed. The two computations are however in agreement once the scheme dependence is taken into account.

5 The $\Delta I = 1/2$ Selection Rule

As discussed in the introduction, in order to restrict the possible values of the input parameters M , $\langle \bar{q}q \rangle$ and $\langle \alpha_s GG/\pi \rangle$ we study the $\Delta I = 1/2$ selection rule which characterizes the $I = 0$ and $I = 2$ amplitudes of the non-leptonic kaon decays.

It is convenient to write the generical amplitude for a kaon to decay into a two pion final state of isospin $I = 1, 2$ as

$$\mathcal{A}_I(K \rightarrow \pi\pi) = A_I \exp i(\delta_I) \quad (5.1)$$

where the phase δ_I comes from the final state interactions in the channel I . From eq. (1.1) and that fact that $\text{Im } \tau \ll 1$ we can write the amplitude A_I as

$$A_I \simeq \frac{\sum_i C_i(\mu) \text{Re} \langle Q_i(\mu) \rangle_I}{\cos \delta_I} \quad (5.2)$$

Experimentally the phases δ_I are obtained in terms of the π - π S-wave scattering length [24] at the m_K scale. The values so derived give to a few degrees uncertainty $\delta_0 \simeq 37^\circ$ and $\delta_2 \simeq -7^\circ$, thus obtaining with good accuracy

$$\begin{aligned} \cos \delta_0 &\simeq 0.8 \\ \cos \delta_2 &\simeq 1.0 . \end{aligned} \quad (5.3)$$

As a consequence the $I = 0$ amplitude includes a 20% enhancement from the rescattering phase.

The existence of $\text{Im } A_I \neq 0$ signal the possible presence of direct CP violation in the $K \rightarrow \pi\pi$ decays, while $\omega \equiv |A_2|/|A_0| \simeq \text{Re } A_2/\text{Re } A_0 = 1/22.2$ represents the $\Delta I = 1/2$ selection rule known experimentally for more than forty years [25]. According to our conventions for the isospin amplitudes we have $\text{Re } A_0 = 3.3 \times 10^{-7}$ GeV and $\text{Re } A_2 = 1.5 \times 10^{-8}$ GeV. Using the notation of eq. (1.1) we can write the CP conserving amplitudes as

$$\begin{aligned} \text{Re } A_0 &= \frac{G_F V_{ud} V_{us}}{\sqrt{2} \cos \delta_0} \sum_i z_i \text{Re} \langle Q_i \rangle_0 \\ \text{Re } A_2 &= \frac{G_F V_{ud} V_{us}}{\sqrt{2} \cos \delta_2} \sum_i z_i \text{Re} \langle Q_i \rangle_2 (1 + \Omega_{\eta+\eta'}) , \end{aligned} \quad (5.4)$$

where $\Omega_{\eta+\eta'} = 0.25 \pm 0.05$ represents the isospin breaking effect due to the π^0 - η - η' mixing. For notational convenience henceforth we identify $\text{Re } A_I$ with A_I .

For a long time the explanation of the $\Delta I = 1/2$ rule has been a mystery. It was found some twenty years ago that perturbative QCD qualitatively worked in the right direction [26, 27]—even though the effect was too small. More recently, it has been shown that non-factorizable contributions and chiral loops also work in the right direction [4, 7, 15]. In a previous paper [12], we argued that the selection rule could be accommodated within the context of the χ QM to $O(p^2)$ with the inclusion of the meson loops. Here we present the complete $O(p^4)$ calculation.

To this order in the chiral expansion there are several constraints to be satisfied. In particular, the stability of the solution as well as its γ_5 -independence restrict the possible values of M . From Figs. 1 and 2, obtained for the central values of the parameters given in appendix C, we see that in order to keep the numerical results for A_0 and A_2 within 20% from their experimental values we must constrain M in the range

$$M = 180 \div 220 \text{ MeV} \quad (5.5)$$

The range thus identified is in good agreement with that found by fitting radiative kaon decays [16]. We notice that the NLO corrections improve the γ_5 -stability, especially for A_2 . Henceforth we will use for the numerical discussion the HV results.

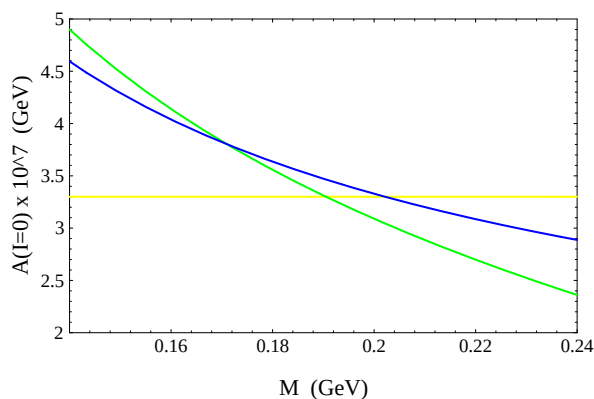


Figure 1: $A_0(K^0 \rightarrow \pi\pi)$ as a function of M in the HV (gray) and NDR (black) schemes respectively. The horizontal line indicates the experimental value.

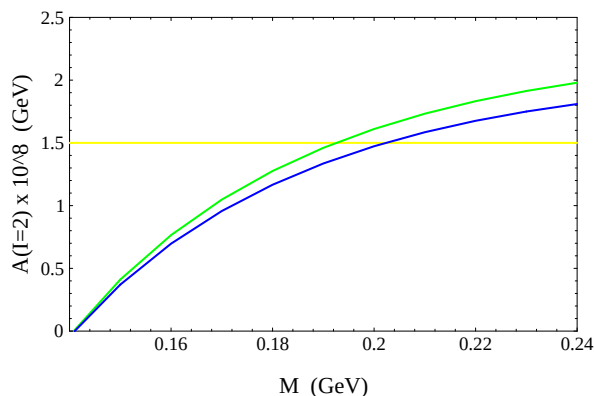


Figure 2: $A_2(K^0 \rightarrow \pi\pi)$ as a function of M in the HV (gray) and NDR (black) schemes respectively. The horizontal line indicates the experimental value.

Given the above range for M , we can study the dependence of the selection rule on the other χ QM parameters. We find that for central values of M and $\Lambda_{QCD}^{(4)}$, namely

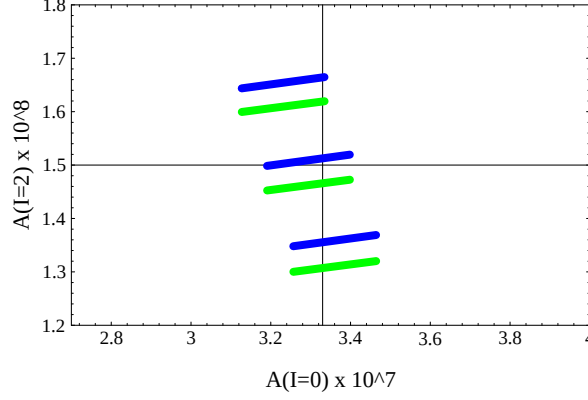


Figure 3: A_0 and A_2 in units of GeV as functions of $\langle\alpha_s GG/\pi\rangle$ (vertical spread) and $\langle\bar{q}q\rangle$ (horizontal spread) for fixed $M = 200$ MeV and central value of $\Lambda_{QCD}^{(4)}$. The cross-hairs indicate the experimental values. Black and gray lines correspond to $m_s(0.8 \text{ GeV}) = 240$ MeV and 200 MeV respectively. The ranges of the gluon and quark condensates are those discussed in the text. The slight slope in the curves is due to the $\langle\bar{q}q\rangle$ dependence in the electroweak operators.

$M = 200$ MeV and $\Lambda_{QCD}^{(4)} = 340$ MeV, the $\Delta I = 1/2$ rule is reproduced with a $\pm 10\%$ accuracy provided

$$\langle\alpha_s GG/\pi\rangle = (334 \pm 4 \text{ MeV})^4, \quad (5.6)$$

and

$$\langle\bar{q}q\rangle = -\left(240^{+30}_{-10} \text{ MeV}\right)^3, \quad (5.7)$$

as it is shown in Fig. 3.

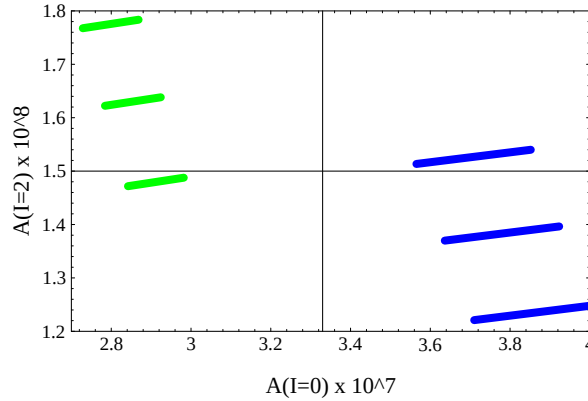


Figure 4: A_0 and A_2 in units of GeV as functions of $\langle\alpha_s GG/\pi\rangle$ (vertical spread) and $\langle\bar{q}q\rangle$ (horizontal spread) in the ranges of eqs. (5.6)–(5.7) for $m_s(\Lambda_\chi) = 220$ MeV and the extreme values of M and $\Lambda_{QCD}^{(4)}$. Black and gray lines correspond to $M = 205$ MeV ($\Lambda_{QCD}^{(4)} = 300$ MeV) and 197 MeV ($\Lambda_{QCD}^{(4)} = 380$ MeV) respectively.

Given the results above, we include the dependence on Λ_{QCD} . Requiring that the rule is finally reproduced with a $\pm 20\%$ approximation, for $\Lambda_{QCD}^{(4)}$ in the range of eq. (1.4) we

obtain a sharp constraint on M (Fig. 4):

$$M = 200^{+5}_{-3} \text{ MeV} . \quad (5.8)$$

In order to exhibit the impact of the NLO corrections we have charted operator by operator the relative weights of the leading order computation, of the one-loop chiral corrections and of the NLO $O(p^4)$ corrections.

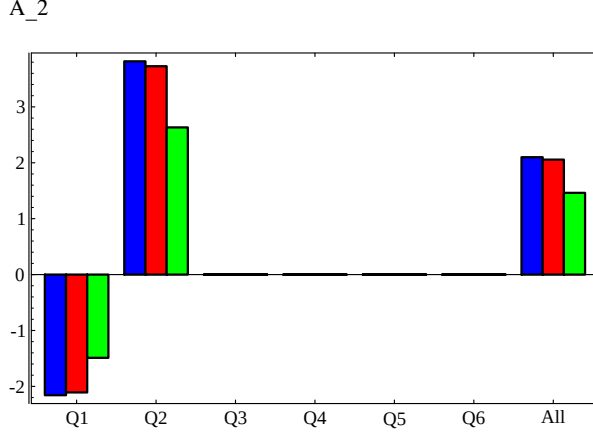


Figure 5: Anatomy of the A_2 amplitude in units of 10^{-8} GeV for central values of the input parameters: LO calculation (black), LO with chiral loops (half-tone), complete NLO result (gray). In the LO case we have taken $f = f_\pi$, whereas in the remaining hystograms the renormalized value $f = 86$ MeV has been used.

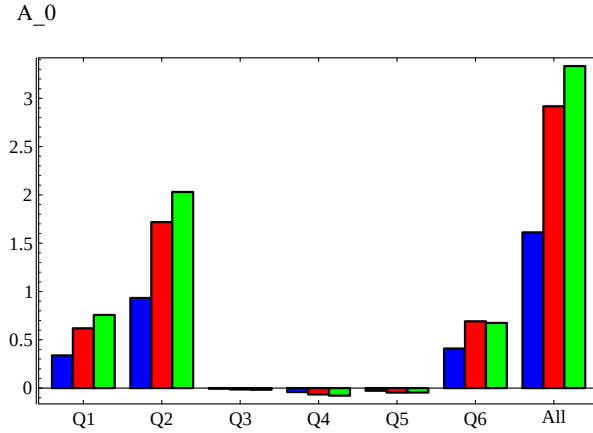


Figure 6: Anatomy of the A_0 amplitude in units of 10^{-7} GeV for central values of the input parameters: LO calculation (black), LO with chiral loops (half-tone), complete NLO result (gray). In the LO case we have taken $f = f_\pi$, whereas in the remaining hystograms the renormalized value $f = 86$ MeV has been used.

As one can see from Figs. 5 and 6, the combined effect of chiral one-loop and NLO χ QM corrections is rather large in the case of A_0 where it leads to an increase of the

amplitude by about 50%, while for the amplitude A_2 chiral loops are negligible and χ QM $O(p^4)$ corrections reduce it by 25%. It is important to remark that the final $O(p^4)$ result goes in the direction indicated by the $\Delta I = 1/2$ rule, that is, of making A_0 larger and A_2 smaller.

Another remark regards the gluon penguin operators. Fig. 6 makes clear that their contribution to the A_0 amplitude is not as large as often claimed. The $O(p^4)$ correction is almost completely accounted for by the chiral loops.

The dependence on m_c is not negligible. For instance, by varying $m_c(m_c)$ from 1.4 to 1.3 GeV, the value of $|\langle \bar{q}q \rangle|^{1/3}$ required in order to fit the rule increases by about 10%.

A convenient way of analyzing the size of hadronic matrix elements in different theoretical approaches is via the B_i factors which quantify the deviation of the hadronic matrix elements in a particular computation from those obtained in the VSA. According to the discussion below eq. (5.1), we define the B_i as

$$B_i^{(0,2)} \equiv \frac{\text{Re} [\langle Q_i \rangle_{0,2}^{\text{model}}]}{\langle Q_i \rangle_{0,2}^{\text{VSA}}}, \quad (5.9)$$

where the VSA matrix elements are by construction real.

In Table 1 we give the B_i coefficients related to the operators Q_{1-6} at $\mu = 0.8$ GeV. The three columns show how the B_i vary from LO (which includes the χ QM gluonic corrections) to the complete NLO result, via the inclusion of meson loops. Their scale dependence has been already discussed in ref. [12], where also the γ_5 -scheme dependence was shown. Since the NLO corrections do not affect the latter we will not repeat the discussion here. Anticipating the results of the next section, we have restricted the values of the input parameters to those required by the fit of the $\Delta I = 1/2$ rule.

	LO	+ χ -loops	+ NLO
$B_1^{(0)}$	4.2	7.8	9.5
$B_2^{(0)}$	1.3	2.4	2.9
$B_1^{(2)} = B_2^{(2)}$	0.60	0.58	0.41
B_3	-0.62	-1.9	-2.3
B_4	1.0	1.6	1.9
$B_5 \simeq B_6$	$1.2 \div 0.72$	$2.0 \div 1.2$	$1.9 \div 1.2$

Table 1: The B_i factors in the χ QM including meson-loops and χ QM NLO corrections. We have taken the gluon condensate at the central value of eq. (5.6), while the ranges given for B_{5-6} correspond to varying the quark condensate according to eq. (5.7). The results shown are given in the HV scheme for $M = 200$ MeV and $f = 86$ MeV, except for the first column where $f = f_\pi$ has been used.

A few comments are in order. The large values for $B_{1,2}^{(0)}$ and the small ones for $B_{1,2}^{(2)}$ reflect the $\Delta I = 1/2$ selection rule, that is well reproduced in our approach. The decrease in the parameter for the operator Q_{5-6} as we increase the value of quark condensate is the consequence of the linear dependence on the quark condensate in the χ QM with respect

to the quadratic one in the VSA. Finally, we confirm the rather large and negative value for B_3 ; the effect of such an unexpected result is discussed in ref. [12].

We conclude by updating in Fig. 7 the “road” to the $\Delta I = 1/2$ rule already presented in ref. [12] where we plot the values of the amplitudes A_0 and A_2 as a functions of the various components for the central values of the input parameters, namely $M = 200$ MeV, $\langle \bar{q}q \rangle = (-240 \text{ MeV})^3$, $\langle \alpha_s GG/\pi \rangle = (334 \text{ MeV})^4$ and $m_s(0.8 \text{ GeV}) = 220$ MeV. This plot includes the NLO $O(p^4)$ corrections (next-to-last step) that we have here computed. Fig. 7 depicts in a suggestive way the decomposition of the fit into its model-dependent components.

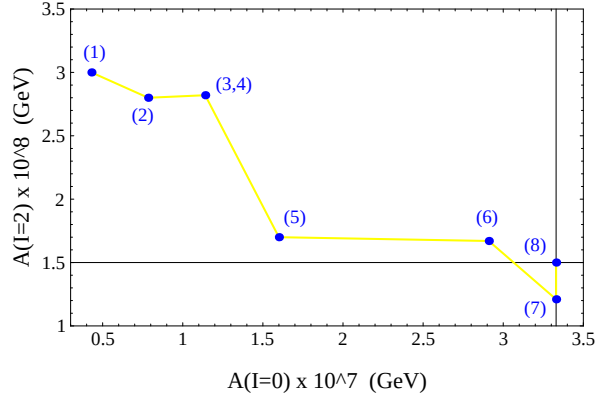


Figure 7: The road to the $\Delta I = 1/2$ rule. The numbered points are discussed in the text.

Point (1) represents the result of free quarks (no QCD, only the operator Q_2 is present) combined with the VSA for the matrix element of Q_2 . Step (2) includes the effects of perturbative QCD renormalization on the operators $Q_{1,2}$. Steps (3,4) show the effect of including gluon and electroweak penguin operators. The latter are responsible for the small shift in A_2 . Therefore, perturbative QCD brings us from (1) to (4). A crucial contribution is given by non-factorizable gluon condensate effects which bring us from (4) to (5)—still remaining at the leading $O(p^2)$ in the chiral expansion. Moving the analysis at the NLO, chiral loops computed on the LO chiral lagrangian lead us from (5) to (6). The NLO $O(p^4)$ corrections characteristic of the χ QM calculated in this paper yield the point (7). Finally, step (8) represents the inclusion of π - η - η' isospin breaking effects which increase A_2 by about 25%.

Let us remark that the exact match with the experimental values should not be taken as a theoretical prediction but rather as the proof of the reproducibility of the experimental rule within our approach. It should be noted once more that the bulk of the effect—up to point (6)—was already included in our previous analysis.

6 The K^0 - $\bar{K}^0$ Mixing Parameter $\hat{B}_K$

The scale-independent parameter $\hat{B}_K$ is defined as the product of the scale dependent $B_K(\mu)$ parameter in

$$\langle \bar{K}^0 | Q_{2S}(\mu) | K^0 \rangle \equiv \frac{4}{3} f_K^2 m_K^2 B_K(\mu), \quad (6.1)$$

and the function $b(\mu)$ introduced in eq. (1.7). Its determination is of crucial relevance in the physics of kaons. We discuss two independent ways by which it can be determined within the χ QM approach.

6.1 $\hat{B}_K$ from a Direct Computation

A “model independent” estimate of $\hat{B}_K$ can be obtained by considering the relationship between the $\Delta S = 2$ matrix element and that of the $\Delta S = 1$ and $\Delta I = 3/2$ amplitude $\mathcal{A}(K^+ \rightarrow \pi^+ \pi^0)$ on the basis of the chiral symmetry [29]

$$\frac{4}{3} f_K^2 m_K^2 \hat{B}_K = \frac{\sqrt{2}}{G_F} \frac{f_\pi}{V_{us}^* V_{ud}} \frac{m_K^2}{m_K^2 - m_\pi^2} \frac{b(\mu)}{z_1(\mu) + z_2(\mu)} \mathcal{A}(K^+ \rightarrow \pi^+ \pi^0). \quad (6.2)$$

Having a model that reproduces the experimental $I = 2$ amplitude, we must subtract from the expression of $\mathcal{A}(K^+ \rightarrow \pi^+ \pi^0)$ all the chiral symmetry breaking components due to chiral loops, NLO corrections, electroweak penguins and $\pi - \eta$ mixing. In this way one obtains in the χ QM approach, on the basis of chiral symmetry alone, the following prediction [12]:

$$\hat{B}_K = \frac{3}{4} b(\mu) \frac{f_\pi^2}{f_K^2} \left[1 + \frac{1}{N_c} (1 - \delta_{\langle GG \rangle}) \right], \quad (6.3)$$

which includes the non-factorizable gluonic corrections.

If we choose for the gluon condensate the value $\langle \alpha_s GG/\pi \rangle = (334 \text{ MeV})^4$, which gives the best fit of $\mathcal{A}(K^+ \rightarrow \pi^+ \pi^0)$ at $O(p^4)$, we obtain at $\mu = 0.8 \text{ GeV}$ and in the HV scheme the prediction

$$\hat{B}_K \simeq 0.47, \quad (6.4)$$

to which all the specific chiral symmetry breaking corrections must be added. Notice that this value is about two times smaller than the LO result depicted in Fig. 8. This is due to the presence of f_π in eq. (6.3) as derived from eq. (6.2), both in the factor f_π^2/f_K^2 and in $\delta_{\langle GG \rangle}$. In our leading order estimate we have instead taken $f = f_K$ ($f_\pi^2/f_K^2 \rightarrow 1$), which we believe is the consistent choice for the tree level evaluation of the K^0 - $\bar{K}^0$ matrix element (as a matter of fact it minimizes the NLO corrections). Thus the leading order χ QM component of the $O(p^4)$ fit amounts to

$$\hat{B}_K \simeq 0.9. \quad (6.5)$$

Adding to eq. (6.3) the meson-loop corrections ($f_\pi \rightarrow f$) and the NLO corrections of eq. (4.19), we find the following formula

$$\hat{B}_K = \frac{3}{4} b(\mu) \frac{f^2}{f_K^2} \left[\left(1 + \frac{1}{N_c} \right) \left(1 + \rho(\mu) + \frac{f_K^2}{f^2} r_{2S}(\mu) \right) \right]$$

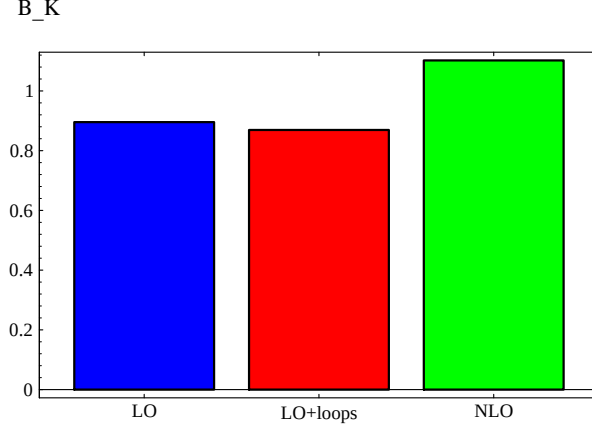


Figure 8: Anatomy of $\hat{B}_K$ for central values of the input parameters in the HV scheme: leading order result (LO), with chiral loops (LO + loops), and complete NLO result (NLO). In the LO case we have taken $f = f_K$, whereas in the remaining histograms the renormalized value $f = 86$ MeV has been used.

$$-\frac{\delta_{\langle GG \rangle}}{N_c} \left(1 + \lambda - \rho(\mu) + \frac{f_K^2}{f^2} r_{2S}(\mu) \right) \Bigg] , \quad (6.6)$$

where the chiral loop effects are included through the function $r_{2S}(\mu)$ of eq. (4.21). The functions ρ and λ are given by

$$\rho(\mu) = \frac{m_s(\mu) + \widehat{m}(\mu)}{M} \left(1 - 6 \frac{M^2}{\Lambda_\chi^2} \right) , \quad (6.7)$$

$$\lambda = \frac{m_K^2}{3M^2} \left(1 - 6 \frac{M^2}{\Lambda_\chi^2} \right) . \quad (6.8)$$

A simpler expression of eq. (6.6) is obtained by replacing f by f_K via the proper χ QM and meson loop renormalizations discussed in subsections 3.2 and 3.3. We then obtain

$$\hat{B}_K = \frac{3}{4} b(\mu) \left[\left(1 + \frac{1}{N_c} \right) (1 + \bar{\tau}_{2S}(\mu)) - \frac{\delta_{\langle GG \rangle}}{N_c} (1 + \lambda - 2 \rho(\mu) + \bar{\tau}_{2S}(\mu)) \right] , \quad (6.9)$$

where $\bar{\tau}_{2S}(\mu)$ is given in eq. (4.22). This corresponds to eq. (3.17) in ref. [17] with the addition of the $O(p^4)$ χ QM corrections here computed.

By varying $\langle \alpha_s GG / \pi \rangle$ in the range of eq. (5.6), M in the range of eq. (5.5) and $\Lambda_{\text{QCD}}^{(4)}$ in that given in eq. (1.4), we find

$$\hat{B}_K = 1.1 \pm 0.2 , \quad (6.10)$$

where the error includes a 8% uncertainty due to the γ_5 -scheme dependence of $b(\mu)$ at $\mu = 0.8$ GeV.

In Fig. 8 we have charted the relative weight of the LO, chiral loop corrections and NLO contributions for central values of the input parameters.. The renormalization of $\hat{B}_K$ from 0.9 to 1.1 amounts to about 20% and it is due to the effect of the NLO corrections

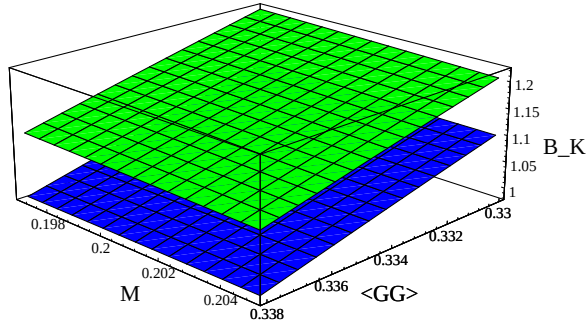


Figure 9: The scale independent parameter $\hat{B}_K$ as a function of M and $\langle GG \rangle \equiv \langle \alpha_s GG / \pi \rangle^{1/4}$ (in GeV) for $m_s(0.8 \text{ GeV}) = 200$ (lower surface) and 240 MeV (upper surface).

that increase the value of the parameter (having chosen $f = f_K$ in the LO estimate minimizes the $O(p^4)$ corrections). This pattern differs from that found in ref. [28] where the NLO corrections act in the opposite direction, reducing the final value. However one has to keep in mind that the anatomy of the various contributions depends on the subtraction prescription used in the renormalization of the meson loops.

To give a direct impression of the variation of $\hat{B}_K$ as we vary the input values, we show in Fig. 9 the overall dependence on $\langle \alpha_s GG / \pi \rangle$, M and m_s . We recall that the NLO result for $\hat{B}_K$ is very little dependent on the value of f used, in the range of eq. (3.12), provided a corresponding fit of the $\Delta I = 1/2$ rule is performed.

This computation has a systematic scale uncertainty—already present at the leading order, and discussed in ref. [17]—due to the poor matching between the long- and the short-distance computations, at variance with the cases of the amplitude A_0 and ε'/ε [13], where the scale dependence of the Wilson coefficients is drastically reduced by the meson loops. This is a consequence of the chiral-loop scale dependence of $a_{S2}(\mu)$ that adds coherently to the running of $b(\mu)$. In this respect higher order corrections do help, since the running of m_s improves the scale stability. Changing the matching scale from 0.8 to 1 GeV has the effect of increasing the value of B_K by about 15%.

As a final remark, in the chiral limit where $m_q = 0$ (but $m_K \neq 0$), the NLO corrections change sign and actually decrease the final value of the parameter; in this case, we find

$$\hat{B}_K(m_q = 0) = 0.47 \pm 0.11. \quad (6.11)$$

This dramatic change qualitatively agrees with what found in ref. [30].

6.2 $\hat{B}_K$ from the K_L - K_S Mass Difference

An alternative and independent way of deriving $\hat{B}_K$ is by computing the $\bar{K}_L$ - K_S mass difference

$$\Delta M_{LS} \equiv m_L - m_S, \quad (6.12)$$

and then comparing with the experimental value

$$\Delta M_{LS}^{\text{exp}} = (3.510 \pm 0.018) \times 10^{-15} \text{ GeV}. \quad (6.13)$$

This computation is made of two parts:

$$\Delta M_{LS} = \Delta M_{SD} + \Delta M_{LD}, \quad (6.14)$$

where ΔM_{SD} accounts for the short-distance contribution due to the $\Delta S = 2$ effective lagrangian in eq. (1.5) and ΔM_{LD} for the long-distance effects generated by the double insertion of the $\Delta S = 1$ chiral lagrangian [17].

By combining eq. (6.13) and eq. (6.14) we can extract the value of $\hat{B}_K$ as

$$\hat{B}_K = b(\mu) \frac{\Delta M_{LS}^{\text{exp}} - \Delta M_{LD}}{C_{S2}(\mu) (4/3) f_K^2 m_K^2}, \quad (6.15)$$

where $C_{S2}(\mu)$ is defined in eq. (1.5).

In a previous paper [17] we have calculated within the χ QM the long-distance contributions to ΔM_{LS} and shown that the net result has the sign opposite to that of the dominant short-distance component, so that the long-distance corrections to the mass difference actually make the prediction smaller¹.

According to eq. (6.15), the value of $\hat{B}_K$ that fits the experimental value of ΔM_{LS} is found to be

$$\hat{B}_K = 1.2 \pm 0.1, \quad (6.16)$$

where the error is obtained by a flat span of the whole range of the input values ($\Lambda_{\text{QCD}}^{(4)}$ included). The effect of changing the matching scale from 0.8 GeV to 1 GeV reduces the absolute value of ΔM_{LD} , corresponding to a decrease of B_K by less than 10%, which is within the error given.

The overlap of the two results (6.10) and (6.16) is rather encouraging and should dispell the concern about the rather large value of $\hat{B}_K$ we find.

Acknowledgments

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A $O(m_q^2)$ Contributions to $\langle Q_6 \rangle$

As pointed out in ref. [14] the matrix element of Q_6 can be written as

$$\begin{aligned} \langle \pi^+ \pi^- | Q_6 | K^0 \rangle &= 2 \langle \pi^- | \bar{u} \gamma_5 d | 0 \rangle \langle \pi^+ | \bar{s} u | K^0 \rangle - 2 \langle \pi^+ \pi^- | \bar{d} d | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle \\ &\quad + 2 \left[\langle 0 | \bar{s} s | 0 \rangle - \langle 0 | \bar{d} d | 0 \rangle \right] \langle \pi^+ \pi^- | \bar{s} \gamma_5 d | K^0 \rangle. \end{aligned} \quad (A.1)$$

¹In the present estimate of ΔM_{LD} we have not used the Gell-Mann-Okubo (GMO) relation (as we did in ref. [17]). As a consequence ΔM_{LD} includes also the single-pole diagrams that vanish in the limit in which the GMO relation holds.

The last line in eq. (A.1) was previously neglected because $\langle \pi^+ \pi^- | \bar{s} \gamma_5 d | K^0 \rangle$ is zero to second order in momentum. However, there are $O(m_q^2)$ terms, obtained when $[\langle 0 | \bar{s} s | 0 \rangle - \langle 0 | \bar{d} d | 0 \rangle] \sim (m_s - m_d)$ is combined with $O(m_q)$ contributions from $\langle \pi^+ \pi^- | \bar{s} \gamma_5 d | K^0 \rangle$, which a priori have to be included in a NLO analysis. A direct calculation gives

$$\langle \pi^+ \pi^- | \bar{s} \gamma_5 d | K^0 \rangle = -i\sqrt{2} \left\{ \frac{\langle \bar{q} q \rangle}{3f^2} + \frac{f_+}{f} \left[\frac{1}{3}(m_s + m_d) + (m_d + m_u) \right] \right\}, \quad (\text{A.2})$$

and we must worry about terms proportional to m_s that can be numerically relevant. However, some parts of this contribution can be rotated away, as we shall explain below. In fact, only the last term proportional to $m_d + m_u$ corresponds to a physical contribution.

The quark condensate term in eq. (A.2) combined with the $m_s - m_d$ component of $\langle 0 | \bar{s} s | 0 \rangle - \langle 0 | \bar{d} d | 0 \rangle$ correspond to a chiral lagrangian term proportional to

$$\text{Tr} \left[\Sigma^\dagger \mathcal{M}_q \lambda_- + \lambda_- \mathcal{M}_q^\dagger \Sigma \right]. \quad (\text{A.3})$$

where $\lambda_\pm = (\lambda_6 \pm \lambda_7)/2$ are the combination of Gell-Mann matrices projecting out $\Delta S = \pm 1$ transitions. This term contains K^0 to vacuum transitions and can therefore be rotated away [14] in agreement with the FKW theorem [31]. In general,

$$\langle 0 | Q_6 | K^0 \rangle = 2 \left[\langle 0 | \bar{s} s | 0 \rangle - \langle 0 | \bar{d} d | 0 \rangle \right] \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle, \quad (\text{A.4})$$

and we have to find out whether there are also terms $O(m_q^2)$ which can be rotated away. To investigate this we do a calculation in the rotated picture of the χQM . Within this picture [14],

$$Q_6 = -8 (F_{(-)})_{\alpha\beta} (\bar{\mathcal{Q}}_L)_\alpha (\mathcal{Q}_R)_\delta (\bar{\mathcal{Q}}_R)_\delta (\mathcal{Q}_L)_\beta, \quad (\text{A.5})$$

where $F_{(-)} = \xi \lambda_- \xi^\dagger$, and the Greek letters are flavor indices for the constituent quark fields $\mathcal{Q}_L = \xi q_L$ and $\mathcal{Q}_R = \xi^\dagger q_R$. Here $\xi \xi = \Sigma$, $L = (1 - \gamma_5)/2$, and $R = (1 + \gamma_5)/2$. To obtain the contribution $O(m_q^2)$ from Q_6 to the chiral lagrangian we have to contract the quark fields in Q_6 to the product of two quark loops with two mass insertions. Within the rotated picture, the mass term reads $\mathcal{L}_{mass} = -\bar{\mathcal{Q}} \widetilde{M}_q \mathcal{Q}$, where

$$\widetilde{M}_q \equiv \xi^\dagger \mathcal{M}_q \xi^\dagger L + \xi \mathcal{M}_q^\dagger \xi R \equiv \widetilde{M}_q^S + \widetilde{M}_q^P \gamma_5. \quad (\text{A.6})$$

where $\widetilde{M}_q^S$ and $\widetilde{M}_q^P$ are defined in an obvious manner to be independent of γ_5 , and $\mathcal{M}_q$ is the current mass matrix. There are three diagrams. First, there is a diagram (A) with two mass insertions in the loop involving the right-handed vertex, and no mass insertion in the second quark loop (which corresponds to the quark condensate divided by two), giving the contribution :

$$\mathcal{L}_A = \frac{1}{2} \langle \bar{q} q \rangle (-iN_c) \text{Tr} \left[F_{(-)} R S(p) \widetilde{M}_q S(p) \widetilde{M}_q S(p) \right], \quad (\text{A.7})$$

where $S(p) = (\gamma \cdot p - M)^{-1}$, and the trace is both in flavor and Dirac spaces. Further there is a diagram (B) with both mass insertions in the loop with the left-handed vertex,

and no mass insertion at the right-handed vertex, and finally a diagram (C) with one mass insertion in both loops. The sum of the chiral lagrangians for the diagrams A,B,C are a linear combination of the terms

$$\hat{\mathcal{L}}_{XY} = 4 \text{Tr} \left[F_{(-)} \widetilde{M}_q^X \widetilde{M}_q^Y \right], \quad (\text{A.8})$$

with $(XY) = (SS), (PP), (SP), (PS)$. We have found that the term

$$\hat{\mathcal{L}}_{PP} = \text{Tr} \left[\lambda_- \mathcal{M}_q^\dagger \Sigma \mathcal{M}_q^\dagger \Sigma + \lambda_- \Sigma^\dagger \mathcal{M}_q \Sigma^\dagger \mathcal{M}_q - \lambda_- \Sigma^\dagger \mathcal{M}_q \mathcal{M}_q^\dagger \Sigma \right], \quad (\text{A.9})$$

contains no K to vacuum transitions and will correspond to physical effects. In the total lagrangian $O(m_q^2)$ we also have terms involving the combinations

$$\hat{\mathcal{L}}_{SS} = \hat{\mathcal{L}}_{PP} + 2\delta\hat{\mathcal{L}}, \quad \hat{\mathcal{L}}_{SP} - \hat{\mathcal{L}}_{PS} = 2\delta\hat{\mathcal{L}}, \quad (\text{A.10})$$

where

$$\delta\hat{\mathcal{L}} = \text{Tr} \left[\lambda_- \Sigma^\dagger \mathcal{M}_q \mathcal{M}_q^\dagger \Sigma \right], \quad (\text{A.11})$$

This term has a K^0 to vacuum transition, and can be rotated away. Therefore, we obtain the total contribution $O(m_q^2)$ from Q_6 as

$$\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C = \left[\frac{\langle \bar{q}q \rangle}{M} \left(2f_+ - 6\frac{M^2}{\Lambda_\chi^2} \right) + f_+^2 f^2 \right] f^2 \hat{\mathcal{L}}_{PP} + G_\delta \delta\hat{\mathcal{L}}, \quad (\text{A.12})$$

where the term proportional to $\delta\hat{\mathcal{L}}$ has no physical consequences.

What we found is that the terms in eq. (A.2) containing the numerical factor $1/3$ correspond to $\delta\hat{\mathcal{L}}$. Thus, the numerically most important terms $\sim m_s^2$ are rotated away. Adding the last term proportional to $m_u + m_d$ in (A.2) to what we obtain from the building blocks in section 2, we obtain the result contained in the quantity γ in (4.15), which agrees with the result in eq. (A.12).

B Computing L_5 and L_8 in the Strong Sector

The higher order (renormalized) counterterms in the strong chiral lagrangian can be determined by using the χ QM. This has been done in [10] using the technique of path integral; here we re-derive some of them in a different and simpler way.

The method consists in computing the Green functions with mesons as external states by using the χ QM lagrangian given in eq. (1.9) and then to compare it with the ones computed using the strong chiral lagrangian². From this comparison the higher order counter terms appearing in the strong chiral lagrangian can be determined. This is also the way we have followed in our previous work [11] to determine the $\Delta S = 1$ weak chiral lagrangian up to $O(p^2)$.

²It should be understood that the computation includes only tree diagrams induced by $O(p^2)$ and $O(p^4)$ terms in the strong chiral lagrangian, since the chiral loop contributions are common for the two approaches.

As an example, we present the determination of L_5 and L_8 . The same method can be extended to determine the other higher counter terms in the strong chiral lagrangian.

The propagator of the meson field can be represented in momentum space as,

$$\frac{i}{p^2 - m_0^2 - \Sigma(p^2)}, \quad (\text{B.1})$$

where $\Sigma(p^2)$ is the self-energy of the meson and m_0 is its tree-level (bare) mass. The self energy $\Sigma(p^2)$ may be organized as an expansion around an arbitrary point in the momentum space μ ,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2) \Sigma'(\mu^2) + \tilde{\Sigma}(p^2), \quad (\text{B.2})$$

where the prime indicates the derivative with respect to p^2 and $\tilde{\Sigma}(p^2)$ denotes the other terms in the expansion. Fixing μ to be the on-shell momentum, and defining the physical mass as the pole of the propagator, eq. (B.1) and eq. (B.2) lead to

$$m_0^2 + \Sigma(m^2) = m^2, \quad (\text{B.3})$$

which relates the physical mass and the bare mass. The wave-function renormalization is then given by

$$Z = [1 - \Sigma'(m^2)]^{-1} = 1 + \Sigma'(m^2) + [\Sigma'(m^2)]^2 + \dots \quad (\text{B.4})$$

Computing the two-point function by means of the χ QM, we get

$$\begin{aligned} \Sigma(p^2) = & -\frac{p^4}{\Lambda_\chi^2} + 6\frac{M(m_u + m_d)}{\Lambda_\chi^2}p^2 - (m_u + m_d)^2 \\ & -\frac{p^6}{10M^2\Lambda_\chi^2} + \frac{(m_u + m_d)}{M\Lambda_\chi^2}p^4 - 6\frac{m_um_d}{\Lambda_\chi^2}p^2 + O(m_q^3). \end{aligned} \quad (\text{B.5})$$

From eq. (B.3) and eq. (B.4), we obtain

$$\begin{aligned} Z_\pi = & 1 - 2\frac{m_\pi^2}{\Lambda_\chi^2} + 6\frac{M(m_u + m_d)}{\Lambda_\chi^2} - 6\frac{m_um_d}{\Lambda_\chi^2} + 2\frac{(m_u + m_d)m_\pi^2}{M\Lambda_\chi^2} - \frac{3m_\pi^4}{10M^2\Lambda_\chi^2} \\ & + \left[2\frac{m_\pi^2}{\Lambda_\chi^2} - 6\frac{M(m_u + m_d)}{\Lambda_\chi^2} \right]^2 + O(m_\pi^6) \end{aligned} \quad (\text{B.6})$$

and

$$\begin{aligned} m_\pi^2 = & (m_\pi^0)^2 - \frac{(m_u + m_d)^2}{\Lambda_\chi^2}B_0^2 + 6\frac{M(m_u + m_d)^2}{\Lambda_\chi^2}B_0 - (m_u + m_d)^2 \\ & + \frac{(m_u + m_d)^3}{M\Lambda_\chi^2}B_0^2 - 6\frac{m_um_d(m_u + m_d)}{\Lambda_\chi^2}B_0 + O(m_\pi^6, m_q^3) \end{aligned} \quad (\text{B.7})$$

with

$$B_0 \equiv -\frac{\langle \bar{q}q \rangle}{f^2} = \frac{m_\pi^2}{(m_u + m_d)}. \quad (\text{B.8})$$

The decay constants f_π and f_K are going to be necessary in determining L_5 . The coupling f_π computed in the chiral quark model is given at $O(p^2, m_q)$ by

$$f_\pi = f \left[1 - \frac{f^2}{\langle \bar{q}q \rangle} \frac{m_\pi^2}{2M} \left(1 - 6 \frac{M^2}{\Lambda_\chi^2} \right) \right]. \quad (\text{B.9})$$

The results for f_π and m_π^2 computed from the strong chiral lagrangian by including the relevant $O(p^4)$ counterterms (but no chiral loops) are,

$$f_\pi = f \left[1 + 4 \frac{m_\pi^2}{f^2} L_5 + \frac{8m_K^2 + 4m_\pi^2}{f^2} L_4 \right], \quad (\text{B.10})$$

and

$$m_\pi^2 = (m_\pi^0)^2 + \frac{8}{f^2} B_0^2 (m_u + m_d)^2 [2L_8 - L_5]. \quad (\text{B.11})$$

Comparing eq. (B.9) and eq. (B.10) implies

$$L_5 = -\frac{f^4}{8\langle \bar{q}q \rangle} \frac{1}{M} \left(1 - 6 \frac{M^2}{\Lambda_\chi^2} \right), \quad (\text{B.12})$$

which is also obtained by a direct calculation within the χ QM (in the rotated picture of [14]). Accordingly, at the same level of approximation, we find

$$L_4 = 0. \quad (\text{B.13})$$

From eq. (B.7) and eq. (B.11) we get at $O(m_q^2)$

$$2L_8 - L_5 = \frac{f^2}{8B_0^2} \left[-\frac{B_0^2}{\Lambda_\chi^2} + 6B_0 \frac{M}{\Lambda_\chi^2} - 1 \right]. \quad (\text{B.14})$$

Eq. (B.14) together with eq. (B.12), yields

$$L_8 = -\frac{N_c}{16\pi^2} \frac{1}{24} - \frac{f^4}{16\langle \bar{q}q \rangle M} \left(1 + \frac{Mf^2}{\langle \bar{q}q \rangle} \right). \quad (\text{B.15})$$

The coupling f_K and m_K could be equally used in determining L_4 , L_5 and L_8 leading to the same result.

C Input Parameters

parameter	value
$\sin^2 \theta_W(m_Z)$	0.23
m_Z	91.187 GeV
m_W	80.33 GeV
m_t^{pole}	175 ± 6 GeV
$m_b(m_b)$	4.4 GeV
$m_c(m_c)$	1.4 GeV
m_s (1 GeV)	178 ± 18 MeV
$m_u + m_d$ (1 GeV)	12 ± 2.5 MeV
$\Lambda_{QCD}^{(4)}$	340 ± 40 MeV
V_{ud}	0.9753
V_{us}	0.221
M	200^{+5}_{-3} MeV
$\langle \bar{q}q \rangle$	$-(240^{+30}_{-10} \text{ MeV})^3$
$\langle \alpha_s GG/\pi \rangle$	$(334 \pm 4 \text{ MeV})^4$
f	86 ± 13 MeV
$f_\pi = f_{\pi^+}$	92.4 MeV
$f_K = f_{K^+}$	113 MeV
$m_\pi = (m_{\pi^+} + m_{\pi^0})/2$	138 MeV
$m_K = m_{K^0}$	498 MeV
m_η	548 MeV
$\cos \delta_0$	0.8
$\cos \delta_2$	1.0

Table 2: Table of the numerical values of the input parameters.

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