

SPHERICALLY SYMMETRIC SOLUTIONS IN MØLLER'S TETRAD THEORY OF GRAVITATION

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ABSTRACT. The general solution of Møller's field equations in case of spherical symmetry is derived. The previously obtained solutions are verified as special cases of the general solution.

PACS number: 40.50.+h

1. INTRODUCTION

Møller [1] modified General Relativity (GR) by constructing a new field theory using Weintzenböck space. His aim was to get a theory free from singularities while retaining the main merits of GR as far as possible. For instance, the principle of general relativity, the principle of equivalence, and the fusion of gravity and mechanics are still valid. Møller's Theory leads to a more satisfactory solution to the problem of defining an energy-momentum complex describing the energy contents of physical systems. This problem has no solution in the framework of gravitational theories based on Riemannian space [2]. Sáez [3] generalized Møller's Theory in a very elegant and natural way into scalar tetradic theories of gravitation. Meyer [4] showed that Møller's Theory is a special case of Poincaré Gauge Theory constructed by Hehl et al. [5].

In an earlier paper [6] the authors examined this theory with regard to the energy-momentum complex. The authors used a spherically symmetric tetrad constructed by Robertson [7] to derive two different spherically symmetric solutions of Møller's field equations. The purpose of the present work is to derive the general solution of Møller's field equations for this tetrad. In Section 2 we will review briefly Møller's Tetrad Theory of Gravitation. The structure of Weintzenböck spaces with spherical symmetry as well as the previously derived solutions of Møller's field equations are reviewed in Section 3. The general solution of Møller's field equations is derived in Section 4. The results are discussed and concluded in Section 5.

2. MØLLER'S TETRAD THEORY OF GRAVITATION

Møller [1] constructed a gravitational theory using Weintzenböck space for its structure. His aim was to get a theory free from singularities while retaining the main merits of GR as far as possible. In his theory the field variables are the 16 tetrad components e_m^μ . Hereafter

Published in General Relativity and Gravitation **26** (1994), 869–876.

we use Latin indices ($mn\dots$) for vector numbers and Greek indices ($\mu\nu\dots$) for vector components. All indices run from 0 to 3. The metric is a derived quantity, given by

$$g^{\mu\nu} := e_m{}^\mu e_m{}^\nu. \quad (2.1)$$

We assume imaginary values for the vector $e_0{}^\mu$ in order to have a Lorentz signature. We note here that, associated with any tetrad field $e_m{}^\mu$ there is a metric field defined uniquely by (2.1), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads $e_m{}^\mu$ leads to a new set of tetrads which also satisfy (2.1).

A central role in Møller's theory is played by the tensor

$$\gamma_{\mu\nu\sigma} := e_{m\mu} e_{m\nu;\sigma}, \quad (2.2)$$

where the semicolon denotes covariant differentiation using the Christoffel symbols. This tensor has close relations to Ricci rotation coefficients and to torsion (cf. the Appendix of Ref. [4]). The tensor $\gamma_{\mu\nu\sigma}$ is invariant only under global Lorentz transformations. Local Lorentz invariance is lost in gravitational theories constructed using Weintzenböck space. These theories admit only the weak form of the principle of equivalence. Møller considered the Lagrangian L to be an invariant constructed from $\gamma_{\mu\nu\sigma}$ and $g_{\mu\nu}$. As he pointed out, the most simple possible independent expressions are

$$L^{(1)} := \Phi_\mu \Phi^\mu, \quad L^{(2)} := \gamma_{\mu\nu\sigma} \gamma^{\mu\nu\sigma}, \quad L^{(3)} := \gamma_{\mu\nu\sigma} \gamma^{\sigma\nu\mu}, \quad (2.3)$$

where Φ_μ is the basic vector defined by

$$\Phi_\mu := \gamma^\nu{}_{\mu\nu}. \quad (2.4)$$

These expressions $L^{(i)}$ in (2.3) are homogeneous quadratic functions in the first order derivatives of the tetrad field components.

Møller considered the simplest case, in which the Lagrangian L is a linear combination of the quantities $L^{(i)}$. That is, the Lagrangian density is given by

$$\mathcal{L}_{\text{Møller}} := (-g)^{1/2} (\alpha_1 L^{(1)} + \alpha_2 L^{(2)} + \alpha_3 L^{(3)}), \quad (2.5)$$

where

$$g := \det(g_{\mu\nu}). \quad (2.6)$$

Møller chose the constants α_i such that his theory gives the same results as GR in the linear approximation of weak fields. According to his calculations, one can easily see that if we choose

$$\alpha_1 = -1, \quad \alpha_2 = \lambda, \quad \alpha_3 = 1 - 2\lambda, \quad (2.7)$$

with λ equals to a free dimensionless parameter of order unity, the theory will be in agreement with GR to the first order of approximation. For $\lambda = 0$, Møller's field equations are identical with Einstein's equations

$$G_{\mu\nu} = -\kappa T_{\mu\nu}. \quad (2.8)$$

For $\lambda \neq 0$, the field equations are given by

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (2.9)$$

$$F_{\mu\nu} = 0, \quad (2.10)$$

where

$$H_{\mu\nu} := \lambda \left[\gamma_{\alpha\beta\mu} \gamma^{\alpha\beta}{}_{\nu} + \gamma_{\alpha\beta\mu} \gamma_{\nu}{}^{\alpha\beta} + \gamma_{\alpha\beta\nu} \gamma_{\mu}{}^{\alpha\beta} + g_{\mu\nu} \left(\gamma_{\alpha\beta\sigma} \gamma^{\sigma\beta\alpha} - \frac{1}{2} \gamma_{\alpha\beta\sigma} \gamma^{\alpha\beta\sigma} \right) \right] \quad (2.11)$$

and

$$F_{\mu\nu} := \lambda \left[\Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\alpha} \left(\gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\nu\mu} \right) + \gamma_{\mu\nu}{}^{\alpha}{}_{;\alpha} \right]. \quad (2.12)$$

Equations (2.10) are independent of the free parameter λ . On the other hand, the term $H_{\mu\nu}$ by which equations (2.9) deviate from Einstein's field equations (2.8) increases with λ , which can be taken of order unity without destroying the first order agreement with Einstein's theory in case of weak fields.

3. SPHERICALLY SYMMETRIC SOLUTIONS IN MØLLER'S THEORY

The structure of Weintzenböck spaces with spherical symmetry has been studied by Robertson [7]. In spherical polar coordinates the four tetrad vectors defining such structure, which admits improper rotations as well, can be written as

$$e_m{}^{\mu} = \begin{pmatrix} A & Dr & 0 & 0 \\ 0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & -\frac{B \sin \phi}{r \sin \theta} \\ 0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\ 0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0 \end{pmatrix}, \quad (3.1)$$

where A , B , and D are functions of r . Since one has to take the vector $e_0{}^{\mu}$ to be imaginary, in order to preserve the Lorentz signature for the metric, the functions A and D have to be taken as imaginary.

Using the tetrad (3.1) to solve Møller's field equations (2.9) and (2.10) we find that equation (2.10) is satisfied identically, and also that $H_{\mu\nu}$ as given by (2.11) vanishes identically. Thus for spherically symmetric exterior solutions, Møller's field equations are reduced to Einstein's field equations of GR, namely

$$G_{\mu\nu} = 0. \quad (3.2)$$

Einstein tensor $G_{\mu\nu}$ may be evaluated using the Riemannian metric derived from (3.1) via the relation (2.11). It is easy to get

$$\begin{aligned} g_{00} &= \frac{B^2 + D^2 r^2}{A^2 B^2}, & g_{10} &= g_{01} = -\frac{Dr}{AB^2}, & g_{11} &= \frac{1}{B^2}, \\ g_{22} &= \frac{r^2}{B^2}, & g_{33} &= \frac{r^2 \sin^2 \theta}{B^2}. \end{aligned} \quad (3.3)$$

The corresponding field equations (3.2) have given rise to equations (5.3)–(5.7) in Ref. [6] which will not be repeated here.

The trivial flat space-time solution for the field equations is obtained by taking

$$A = i, \quad B = 1, \quad D = 0. \quad (3.4)$$

A first non-trivial solution can be obtained by taking $D = 0$ and solving for A and B . This is the case studied by Møller [1], where he obtained the solution

$$A = i \frac{(1 + m/2r)}{(1 - m/2r)}, \quad B = (1 + m/2r)^{-2}, \quad D = 0. \quad (3.5)$$

A second non-trivial solution can be obtained by taking $A = i$, $B = 1$, $D \neq 0$ and solving for D . In this case the resulting field equations can be integrated directly [6]. The result is

$$A = i, \quad B = 1, \quad D = i \left(\frac{2m}{r^3} \right)^{1/2}, \quad (3.6)$$

where m is a constant of integration which can be identified with the mass of the source generating the field [6].

4. GENERAL SOLUTION OF MØLLER'S FIELD EQUATIONS

Mikhail and Wanas [8] proposed a generalized field theory based on Weintzenböck space, which has close formal similarities with Møller's Theory [9]. Wanas [10] sought spherically symmetric solutions using the tetrad (3.1). Mazumder and Ray [11] completely integrated the field equations of Mikhail-Wanas Theory for the tetrad (3.1) by a suitable change of variables. Due to the formal similarities between Mikhail-Wanas Theory and Møller's Theory, one expects a method for solving the field equations in one theory to be applicable in the other.

In fact, it is clear that the tetrad (3.1) will have a simple r dependence if we divide the first spatial component of every vector by r . This can be achieved by the coordinate transformation

$$r \rightarrow \rho = \ln r. \quad (4.1)$$

This transformation along with the substitutions

$$A = i\mathcal{A}, \quad B = \exp(\rho)\mathcal{B}, \quad D = i\mathcal{D}, \quad (4.2)$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ are real functions of ρ , turn the tetrad (3.1) into the simple form

$$e_m{}^\mu = \begin{pmatrix} i\mathcal{A} & i\mathcal{D} & 0 & 0 \\ 0 & \mathcal{B} \sin \theta \cos \phi & \mathcal{B} \cos \theta \cos \phi & -\mathcal{B} \frac{\sin \phi}{\sin \theta} \\ 0 & \mathcal{B} \sin \theta \sin \phi & \mathcal{B} \cos \theta \sin \phi & \mathcal{B} \frac{\cos \phi}{\sin \theta} \\ 0 & \mathcal{B} \cos \theta & -\mathcal{B} \sin \theta & 0 \end{pmatrix}. \quad (4.3)$$

The Riemannian metric associated with the tetrad (4.3) is given by

$$\begin{aligned} g_{00} &= \frac{\mathcal{D}^2 - \mathcal{B}^2}{\mathcal{A}^2 \mathcal{B}^2}, & g_{10} &= g_{01} = -\frac{\mathcal{D}}{\mathcal{A} \mathcal{B}^2}, & g_{11} &= \frac{1}{\mathcal{B}^2}, \\ g_{22} &= \frac{1}{\mathcal{B}^2}, & g_{33} &= \frac{\sin^2 \theta}{\mathcal{B}^2}. \end{aligned} \quad (4.4)$$

The tetrad $e_m{}^\mu$ and the metric $g_{\mu\nu}$ as given by (4.3) and (4.4) do not depend on ρ explicitly, in contrast to their explicit dependence on r as given by (3.1) and (3.3). The corresponding field equations (3.2) have given rise to the following differential equations:

$$3\mathcal{B}^2 \mathcal{B}_\rho^2 - \mathcal{B}^4 + 2\mathcal{B} \mathcal{D} \mathcal{B}_\rho \mathcal{D}_\rho - 5\mathcal{D}^2 \mathcal{B}_\rho^2 + 2\mathcal{B} \mathcal{D}^2 \mathcal{B}_{\rho\rho} - 2\mathcal{B}^3 \mathcal{B}_{\rho\rho} = 0, \quad (4.5)$$

$$\mathcal{A} \mathcal{B}^2 \mathcal{B}_\rho^2 + 2\mathcal{B}^3 \mathcal{A}_\rho \mathcal{B}_\rho - \mathcal{A} \mathcal{B}^4 + 2\mathcal{A} \mathcal{B} \mathcal{D} \mathcal{B}_\rho \mathcal{D}_\rho - 5\mathcal{A} \mathcal{D}^2 \mathcal{B}_\rho^2 + 2\mathcal{A} \mathcal{B} \mathcal{D}^2 \mathcal{B}_{\rho\rho} = 0, \quad (4.6)$$

$$\begin{aligned} &5\mathcal{A}^2 \mathcal{B} \mathcal{D} \mathcal{B}_\rho \mathcal{D}_\rho - \mathcal{A}^2 \mathcal{B}^2 \mathcal{D} \mathcal{D}_{\rho\rho} - 3\mathcal{A} \mathcal{B} \mathcal{D}^2 \mathcal{A}_\rho \mathcal{B}_\rho - 5\mathcal{A}^2 \mathcal{D}^2 \mathcal{B}_\rho^2 \\ &+ 2\mathcal{A}^2 \mathcal{B} \mathcal{D}^2 \mathcal{B}_{\rho\rho} - 2\mathcal{B}^2 \mathcal{D}^2 \mathcal{A}_\rho^2 - \mathcal{A}^2 \mathcal{B}^3 \mathcal{B}_{\rho\rho} + 3\mathcal{A} \mathcal{B}^2 \mathcal{D} \mathcal{A}_\rho \mathcal{D}_\rho \\ &+ \mathcal{A} \mathcal{B}^2 \mathcal{D}^2 \mathcal{A}_{\rho\rho} + 2\mathcal{B}^4 \mathcal{A}_\rho^2 - \mathcal{A}^2 \mathcal{B}^2 \mathcal{D}_\rho^2 - \mathcal{A} \mathcal{B}^4 \mathcal{A}_{\rho\rho} + \mathcal{A}^2 \mathcal{B}^2 \mathcal{B}_\rho^2 = 0, \end{aligned} \quad (4.7)$$

where the subscript ρ refers to differentiation with respect to ρ .

By eliminating the function $\mathcal{D}$ between equations (4.5) and (4.6) we get

$$\mathcal{A} \mathcal{B}^2 \mathcal{B}_\rho^2 - \mathcal{A} \mathcal{B}^3 \mathcal{B}_{\rho\rho} - \mathcal{B}^3 \mathcal{A}_\rho \mathcal{B}_\rho = 0. \quad (4.8)$$

Equation (4.8) can be written in the form

$$\frac{\mathcal{A}_\rho}{\mathcal{A}} = \frac{\mathcal{B}_\rho}{\mathcal{B}} - \frac{\mathcal{B}_{\rho\rho}}{\mathcal{B}_\rho}. \quad (4.9)$$

This can be integrated directly. The result is

$$\mathcal{A} = \frac{C_1 \mathcal{B}}{\mathcal{B}_\rho}, \quad (4.10)$$

where C_1 is an arbitrary constant. Equation (4.5) can be written in the form

$$a(\rho) \frac{d}{d\rho} \mathcal{D}^2 + b(\rho) \mathcal{D}^2 + c(\rho) = 0, \quad (4.11)$$

where

$$\begin{aligned} a(\rho) &= \mathcal{B}\mathcal{B}_\rho, \\ b(\rho) &= 2\mathcal{B}\mathcal{B}_{\rho\rho} - 5\mathcal{B}_\rho^2, \\ c(\rho) &= 3\mathcal{B}^2\mathcal{B}_\rho^2 - \mathcal{B}^4 - 2\mathcal{B}^3\mathcal{B}_{\rho\rho}. \end{aligned} \quad (4.12)$$

The general solution of equation (4.11) is

$$\mathcal{D}^2 = C_2 \exp \left\{ - \int \frac{b(\rho)}{a(\rho)} d\rho \right\} - \exp \left\{ - \int \frac{b(\rho)}{a(\rho)} d\rho \right\} \times \int \frac{c(\rho)}{a(\rho)} \exp \left\{ \int \frac{b(\rho)}{a(\rho)} d\rho \right\} d\rho, \quad (4.13)$$

where C_2 is an arbitrary constant. The integrals in equation (4.13) can be evaluated in closed form. The result is

$$\mathcal{D}^2 = \frac{C_2\mathcal{B}^5}{\mathcal{B}_\rho^2} - \frac{\mathcal{B}^2(\mathcal{B}^2 - \mathcal{B}_\rho^2)}{\mathcal{B}_\rho^2}. \quad (4.14)$$

Finally, direct substitution of $\mathcal{A}$ and $\mathcal{D}^2$, as given by (4.10) and (4.14) in equation (4.7), shows that it is satisfied without any further restrictions on the two constants C_1 and C_2 or on the function $\mathcal{B}$. So the general solution of (4.5)–(4.7) is given by (4.10) and (4.14).

In terms of the original variables the general solution can be written as

$$\begin{aligned} A &= \frac{iC_1}{1 - rB'/B}, \\ D^2 &= -\frac{C_2B^3}{r^3(1 - rB'/B)^2} + \frac{BB'}{r} \frac{(2 - rB'/B)}{(1 - rB'/B)^2}. \end{aligned} \quad (4.15)$$

The previously obtained solutions can be verified as special cases of the general solution by suitable choice of C_1 , C_2 , and B . The choice

$$C_1 = 1, \quad C_2 = 2m, \quad B = (1 + m/2r)^{-2} \quad (4.16)$$

gives rise to the solution (3.5). On the other hand, the choice

$$C_1 = 1, \quad C_2 = 2m, \quad B = 1 \quad (4.17)$$

gives rise to the solution (3.6).

5. CONCLUDING REMARKS

The general solution of Møller's field equations for the tetrad which admits spherical symmetry and improper rotations has been obtained. The previously obtained solutions have been verified as special cases of the general solution.

The general solution has been found to contain an arbitrary function and two constants. Hence, Møller's field equations do not fix the tetradic geometry in case of spherical symmetry, up to any finite number of arbitrary constants. It is to be noted here that in the cosmological

case [12] and in the stationary axisymmetric case [13], it was proved that Møller's field equations do not fix the tetrad field. The general solution was not obtained in these cases. Sáez [3] generalized Møller's Theory in a very natural way into scalar tetradic theories of gravitation. An important question in these theories is whether the field equations fix the tetradic geometry in the case of spherical symmetry. This question was discussed at length by Sáez [14], but without giving a conclusive answer. In the light of the present work, it will be a major advantage of Sáez's theories over Møller's Theory if one can answer this question in the affirmative. This needs more investigations before arriving at a final answer.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Dr. M. Melek, Cairo University, for his stimulating discussions. They would also like to thank one of their referees for his invaluable comments and suggestions as well as for his independent check of the validity of the general solution given in this paper.

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